A remark on the superhedging theorem under transaction costs

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Summary. The hedging theorem of [3] describes the initial endowments necessary for the super-replication of a given contingent claim in a model with transaction costs, assuming the continuity of the price process. We demonstrate that this theorem may fail if the price process is discontinuous.

1 Model specification

In [2], [3] and [4] the authors describe initial portfolios which allow to hedge a given contingent claim in a market model with proportional transaction costs. All these articles assume the continuity of the price process. We show that this hypothesis is essential for the validity of the theorem. We give a short description of the model and refer to [3] for more detailed information.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a stochastic basis with finite time horizon T and $\mathcal{F}_0$ trivial. Let S be a d-dimensional semimartingale with strictly positive components describing the price evolution of d assets quoted in some reference asset (traded or not).

For any d-dimensional process G we define $\widehat{G}$ as

$$\widehat{G}^i_t := \frac{G^i_t}{S^i_t}, \qquad 1 \leqslant i \leqslant d.$$

Let (λ^{ij}) be a $d \times d$ matrix with 0 diagonal and nonnegative entries representing the proportional transaction costs: each time we transfer 1 unit of

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wealth from asset i to asset j, our position in asset j increases by 1 and our position in asset j decreases by $(1+\lambda^{ij})$. The value of a portfolio in the d assets can be represented by elements of $\mathbb{R}^d$. We define the polyhedral cone

$$-M := \Big\{ x \in \mathbb{R}^d : \text{ there is } a^{ij} \geqslant 0,\ 1 \leqslant i,j \leqslant d \quad \text{such that } x^i = \sum_{j=1}^d a^{ji} - \sum_{j=1}^d a^{ij} \Big\}.$$

This is the set of positions which can be obtained from 0 by making transfers (described by the a^{ij}) from asset i to asset j.

The *solvency cone* $K := M + \mathbb{R}^d_+$ is then the positions from which making a suitable transfer we may arrive at a position in $\mathbb{R}^d_+$. It induces a partial order on $\mathbb{R}^d$: $x \preceq y \iff y - x \in K$. The reader can check that if not all of the λ^{ij} are 0 then $K = M$.

We also introduce

$$\widehat{K}_t(\omega) := \big\{ x \in \mathbb{R}^d : \ \big(x^1 S^1_t(\omega), \dots, x^d S^d_t(\omega)\big) \in K \big\}.$$

The value of the agent's position at time t in asset i is supposed to follow the equation

$$V^i = V^i(v, B) = v^i + \widehat{V}^i_- \cdot S^i + B^i,$$

where $v \in \mathbb{R}^d$ is an initial position, B is the agent's strategy, $\cdot$ denotes stochastic integration. We suppose that B is an adapted process with bounded variation such that all its increments lie in $-M$. This condition tries to grasp the idea of self-financing portfolio.

The physical quantity $\widehat{V}^i_t$ of asset i at time t is found to be equal to

$$\widehat{V}^i_t(v, B) = v^i / S^i_0 + (1/S^i) \cdot B^i_t, \qquad 1 \leqslant i \leqslant d. \tag{1}$$

We call a strategy B *admissible*, if there exists $\kappa > 0$ such that

$$-\kappa S_t \preceq V^{v,B}_t, \qquad t \in [0, T].$$

The set of admissible strategies is denoted by $\mathcal{B}_b$. L^0 denotes the set of d-dimensional random variables. L^0_b is the set of random variables U for which there is $\kappa > 0$ with $-\kappa S_T \preceq U$. We define the set of contingent claims which can be super-replicated from v as

$$A^v := \{ U \in L^0_b : \text{there exists } B \in \mathcal{B}_b \text{ such that } V_T(v, B) \succeq U \}.$$

We also introduce

$$\hat{A}^v := \{ V \in L^0 : \text{there exists } U \in A^v \text{ such that } V^i S^i_T = U^i,\ 1 \leqslant i \leqslant d \}.$$

Remark 1. If $M = K$ and $V_T(v, B) \succeq U$ we can always modify B to B' by adding a last transfer (a $-K = -M$-valued random variable) at time T such that $V_T(v, B') = U$. Hence in this case

$$\hat{A}^v = \big\{ \widehat{V}_T(v, B) : \ B \in \mathcal{B}_b \big\}.$$

For any cone $C \subset \mathbb{R}^d$ we define its positive dual cone as

$$C^* := \{x \in \mathbb{R}^d : \ xw \geqslant 0 \text{ for all } x \in C\}.$$

We now define the set of dual variables which will figure in the hedging theorem: $\mathcal{D}$ denotes the set of martingales Z such that $\widehat{Z}_t(\omega) \in K^*$ for all $0 \leqslant t \leqslant T$ and for almost all ω.

The set of initial positions allowing to hedge a given $H \in L^0_b$ is defined as

$$\Gamma_H := \{v \in \mathbb{R}^d : \ H \in A^v\}.$$

In the framework presented above, we mean the following assertion by hedging theorem:

$$\Gamma_H = \big\{v \in \mathbb{R}^d : \ \mathbb{E}\big[\widehat{Z}_T H\big] \leqslant \widehat{Z}_0 v, \ Z \in \mathcal{D}\big\}. \tag{$*$}$$

In [3] $(*)$ is shown in the case where int $K^* \neq \varnothing$, $S^1 \equiv 1$, S is continuous and there exists $\mathbb{P}' \sim \mathbb{P}$ such that S is a $\mathbb{P}'$-martingale. One can relax the hypothesis on the existence of an equivalent martingale measure, see [4]. On the other hand, in section 2 we demonstrate that the continuity assumption can not be dropped.

We now recall a notion of convergence which has proved to be useful in investigations related to arbitrage theory, see [1]. We say that the sequence ζ_n of random variables is Fatou convergent to ζ if there is $\kappa > 0$ with $-\kappa\mathbf{1} \preceq \zeta_n$ for all n and $\zeta_n \to \zeta$ a.s., here $\mathbf{1}$ denotes the vector all of whose components are 1.

Lemma 1. *If* $(*)$ *holds then the set* $\hat{A}^0$ *is Fatou closed.*

Proof. Let us take any sequence $\zeta_n \in \hat{A}^0$ such that for each n

$$-\kappa\mathbf{1} \preceq \zeta_n,$$

for some $\kappa > 0$. Let us suppose that $\zeta_n \to \zeta$ almost surely. As $\zeta_n \in \hat{A}^0$, $(*)$ implies that

$$\mathbb{E}[Z_T\zeta_n] \leqslant 0, \qquad Z \in \mathcal{D}.$$

By the Fatou-lemma we get that

$$\mathbb{E}[Z_T\zeta] \leqslant 0, \qquad Z \in \mathcal{D},$$

and $(*)$ guarantees $\zeta \in \hat{A}^0$.

2 The counterexample

We claim that there is a bounded martingale S such that the corresponding $\hat{A}^0$ is not Fatou closed, hence Lemma 1 contradicts $(*)$.

We take independent random variables $\eta, \xi_i,\ i \geqslant 1$, with distribution

$$\mathbb{P}\{\eta = 1\} = \mathbb{P}\{\eta = -1\} = 1/2,$$
$$\mathbb{P}\{\xi_i = a_i\} = \exp\bigl(-2^{-i}\bigr), \qquad \mathbb{P}\{\xi_i = -1\} = 1 - \exp\bigl(-2^{-i}\bigr),$$

where $0 < a_i < 1$ is such that $\mathbb{E}[\xi_i] = 0$.

We consider a two-asset model on the interval $[0,1]$, $S^1 \equiv 1$ constant,

$$S^2_t(\omega) = 2 + \sum_{i=1}^{\infty} 2^{-i}\xi_i(\omega) I_{[1-1/(i+1),1]}(t) + 2^{-1}a_1\eta(\omega)I_{\{1\}}(t).$$

Clearly, $1/2 < S^2 < 7/2$ is a martingale with respect to its natural filtration: the convergence of the infinite sum follows from $|\xi_i| \leqslant 1$. Let $\lambda^{12} = \lambda^{21} = 1/2$. We see at once that $(-1, 3/2)$ and $(3/2, -1)$ generate the cone K, $\operatorname{int} K^* \neq \varnothing$ and $K = M$. We introduce $e^1 := (1,0),\ e^2 := (0,1)$.

The random variable

$$\zeta := I_{\{\xi_i = a_i,\ i \geqslant 1\}}\left(e^1 - \frac{3}{2S^2_{1-}}\, e^2\right)$$

is in the Fatou closure of $\hat{A}^0$: let us take the strategy B_n which consists of effectuating a portfolio change $I_{\{\xi_j = a_j,\ 1 \leqslant j \leqslant n\}}(e^1 - (3/2)e^2) \in -K$ at time $1 - 1/n$ and otherwise doing nothing.

$$\widehat{V}_1(0, B_n) := I_{\{\xi_j = a_j,\ 1 \leqslant j \leqslant n\}}\left(e^1 - \frac{3}{2S^2_{1-1/n}}\, e^2\right) \longrightarrow \zeta$$

almost surely and this sequence is uniformly bounded. In order to check that $\zeta \notin \hat{A}^0$ we notice that the event

$$D := \{\omega \in \Omega :\ \xi_i(\omega) = a_i,\ i \geqslant 1;\ \eta(\omega) = -1\}$$

has positive probability:

$$\mathbb{P}(D) = \frac{1}{2}\exp\left(-\sum_{i=1}^{\infty} 2^{-i}\right) > 0.$$

On D the trajectories of $S^2_\cdot$ increase on $[0,1)$ to S^2_{1-} and jump downwards at the terminal point. Thus, for $\omega \in D,\ t \in [0,1]$, we have that

$$-\widehat{K}_t(\omega) \subset J(\omega),$$

where $J(\omega) := \operatorname{cone}\{w_1, w_2\} \setminus \mathbb{R}_+ w_2$ with

$$w_1 := \bigl(3/(2S^2_0), -1\bigr) = (3/4, -1), \qquad w_2 = \bigl(1, -3/\bigl(2\sup_{t\in[0,1]} S^2_t\bigr)\bigr).$$

As J is a convex cone, formula (1) entails that on D we have $\widehat{V}_1(0,B) \in J$ for any admissible B while ζ takes values on the ray $\mathbb{R}_+ w_2 \setminus \{0\}$, hence $\widehat{V}_1(0,B) = \zeta$ is not possible. In view of Remark 1 we may conclude that $\zeta \notin \hat{A}^0$.

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