A Remark on Hypercontractivity and Tail Inequalities for the Largest Eigenvalues of Random Matrices

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Summary. We point out a simple argument relying on hypercontractivity to describe tail inequalities on the distribution of the largest eigenvalues of random matrices at the rate given by the Tracy–Widom distribution. The result is illustrated on the known examples of the Gaussian and Laguerre unitary ensembles. The argument may be applied to describe the generic tail behavior of eigenfunction measures of hypercontractive operators.

Introduction

Let $M = M^N$ be a random matrix from the Gaussian Unitary Ensemble (GUE), that is, with distribution

$$\mathbb{P}(\mathrm{d}M) = Z_N^{-1} \exp\bigl(-2N\,\mathrm{Tr}\bigl(M^2\bigr)\bigr)\,\mathrm{d}M$$

where $\mathrm{d}M$ is Lebesgue measure on the space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices.

Denote by $\lambda_1^N, \ldots, \lambda_N^N$ the (real) eigenvalues of M^N. Wigner's theorem indicates that the mean spectral measure $m^N = \mathbb{E}[(1/N)\sum_{i=1}^N \delta_{\lambda_i^N}]$ converges weakly to the semicircle law $\sigma(\mathrm{d}x) = (2/\pi)\sqrt{1-x^2}\,\mathbf{1}_{\{|x|\leqslant 1\}}\,\mathrm{d}x$ (cf. [23]).

The largest eigenvalue $\lambda_{\max}^N = \max_{1\leqslant i\leqslant N} \lambda_i^N$ may be shown to converge almost surely to the right endpoint of the support of the semicircle law, that is 1 with the normalization chosen here. Fluctuations of $\lambda_{\max}^N$ around 1 gave rise to one main achievement due to C. A. Tracy and H. Widom in the recent developments on random matrices. Namely, they showed that fluctuations take place at the rate $N^{2/3}$ and that $N^{2/3}(\lambda_{\max}^N - 1)$ converges weakly to the so-called Tracy–Widom distribution [TW] (cf. [5]). Universality of the Tracy–Widom distribution is conjectured, and has been settled rigorously for large classes of Wigner matrices by A. Soshnikov [19]. For the Laguerre ensemble and Wishart matrices, see [11, 12, 20]. Large deviations for $\lambda_{\max}^N$ of the GUE are described in [3].

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For fixed N, as a Lipschitz function of the Gaussian entries of M^N, the largest eigenvalue $\lambda_{\max}^N$ satisfies the concentration inequality around its mean

$$\mathbb{P}\{|\lambda_{\max}^N - \mathbb{E}(\lambda_{\max}^N)| \geqslant r\} \leqslant 2\,\mathrm{e}^{-2Nr^2} \tag{1}$$

for every $r \geqslant 0$ (cf. [15]). This result however does not yield the fluctuation rate $N^{2/3}$ that requires more refined tools, relying usually on delicate Plancherel–Rotach asymptotics for Hermite polynomials involving the Airy function. The aim of this note is actually to point out a simple argument, based on hypercontractivity, to reach the normalization $N^{2/3}$ and to recover tail inequalities for the largest eigenvalues of some invariant ensembles of interest.

The starting point is the well-known fact (see [17, 5]) that the distribution of the eigenvalues $(\lambda_1^N, \dots, \lambda_N^N)$ of the GUE has density

$$\frac{1}{Z} \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2\, \mathrm{e}^{-2N|x|^2}, \qquad x = (x_1, \dots, x_N) \in \mathbb{R}^N, \tag{2}$$

with respect to Lebesgue measure on $\mathbb{R}^N$ (where Z is the normalization factor). Denote by h_k, $k \in \mathbb{N}$, the normalized Hermite polynomials with respect to the standard normal distribution γ on $\mathbb{R}$. Since, for each k, h_k is a polynomial function of degree k, up to a constant depending on N, the Vandermonde determinant $\prod_{1 \leqslant i < j \leqslant N}(x_i - x_j)$ of (2) is easily seen to be equal to

$$\det\big(h_{i-1}(x_j)\big)_{1 \leqslant i,j \leqslant N}\,.$$

Recall now the mean spectral measure $m^N = \mathbb{E}[(1/N)\sum_{i=1}^N \delta_{\lambda_i^N}]$. If f is a bounded measurable real-valued function on $\mathbb{R}$,

$$\int_{\mathbb{R}} f\,\mathrm{d}m^N = \int_{\mathbb{R}^N} \frac{1}{N} \sum_{i=1}^N f\left(\frac{x}{2\sqrt{N}}\right) \det{}^2\big(h_{i-1}(x_j)\big)_{1 \leqslant i,j \leqslant N}\, \mathrm{e}^{-|x|^2/2}\, \frac{\mathrm{d}x}{Z}\,.$$

Expanding the determinant and using the orthogonality properties of the Hermite polynomials shows that (cf. [17, 5], ...)

$$\int_{\mathbb{R}} f\,\mathrm{d}m^N = \int_{\mathbb{R}} f\left(\frac{x}{2\sqrt{N}}\right) \frac{1}{N} \sum_{k=0}^{N-1} h_k^2(x)\,\gamma(\mathrm{d}x). \tag{3}$$

In particular thus, for every $\varepsilon \geqslant 0$,

$$\mathbb{P}\{\lambda_{\max}^N \geqslant 1 + \varepsilon\} \leqslant N m^N\big([1+\varepsilon, \infty)\big) = \int_{2\sqrt{N}(1+\varepsilon)}^{\infty} \sum_{k=0}^{N-1} h_k^2(x)\,\gamma(\mathrm{d}x).$$

Now, by Hölder's inequality, for every $r > 1$, and every $k = 0, \dots, N-1$,

$$\int_{2\sqrt{N}(1+\varepsilon)}^{\infty} h_k^2(x)\,\gamma(\mathrm{d}x) \leqslant \gamma\Big(\big[2\sqrt{N}(1+\varepsilon),\infty\big)\Big)^{1-(1/r)}\bigg(\int |h_k(x)|^{2r}\,\gamma(\mathrm{d}x)\bigg)^{1/r}$$
$$\leqslant \mathrm{e}^{-2N(1+\varepsilon)^2(1-1/r)}\bigg(\int |h_k(x)|^{2r}\,\gamma(\mathrm{d}x)\bigg)^{1/r}.$$

Consider now the number operator $\mathcal{L}f = f'' - xf'$ with eigenfunctions h_k and corresponding eigenvalues $-k$, $k \in \mathbb{N}$. The associated semigroup $P_t = \mathrm{e}^{t\mathcal{L}}$ satisfies the celebrated hypercontractivity property put forward by E. Nelson [18]

$$\|P_t f\|_q \leqslant \|f\|_p$$

for every $1 < p < q < \infty$ and $t > 0$ such that $\mathrm{e}^{2t} \geqslant (q-1)/(p-1)$ (cf. [2]). Norms are understood here with respect to γ. Since $P_t h_k = \mathrm{e}^{-kt} h_k$, it follows that for every $r > 1$ and $k \geqslant 0$,

$$\|h_k\|_{2r} \leqslant (2r-1)^{k/2}.$$

Hence,

$$\int_{2\sqrt{N}(1+\varepsilon)}^{\infty} \sum_{k=0}^{N-1} h_k^2(x)\,\gamma(\mathrm{d}x) \leqslant \mathrm{e}^{-2N(1+\varepsilon)^2(1-1/r)} \sum_{k=0}^{N-1} (2r-1)^k$$
$$\leqslant \frac{1}{2(r-1)}\,\mathrm{e}^{-2N(1+\varepsilon)^2(1-1/r)+N\log(2r-1)}.$$

Optimizing in $r \to 1$ then shows that, for $0 < \varepsilon \leqslant 1$,

$$\mathbb{P}\big\{\lambda_{\max}^N \geqslant 1+\varepsilon\big\} \leqslant C\varepsilon^{-1/2}\,\mathrm{e}^{-cN\varepsilon^{3/2}} \tag{4}$$

for some numerical values $C, c > 0$.

The same method yields that for $p = [tN^{2/3}]$ (integer part), $t > 0$,

$$Na_p = \mathbb{E}\bigg(\sum_{i=1}^{N} \big(\lambda_i^N\big)^{2p}\bigg) = \frac{1}{\big(2\sqrt{N}\big)^{2p}} \int_{\mathbb{R}} x^{2p} \sum_{k=0}^{N-1} h_k^2(x)\,\gamma(\mathrm{d}x) \leqslant Ct^{-1}N^{1/3}\mathrm{e}^{ct^3} \tag{5}$$

(that may be used to recover (4)).

Besides the polynomial factors in front of the exponential, the preceding bounds (4) and (5) indeed describe the rate $N^{2/3}$ in the fluctuations of $\lambda_{\max}^N$. It does not seem however that one can get rid of these polynomial factors by the preceding hypercontractivity method, that might appear too naive to this task. The optimal bound on the (even) moments a_p of the mean spectral measure may be obtained from the classical recurrence formula (cf. [10, 9])

$$a_p = \frac{2p-1}{2p+2}\,a_{p-1} + \frac{2p-1}{2p+2} \times \frac{2p-3}{2p} \times \frac{p(p-1)}{4N^2}\,a_{p-2} \tag{6}$$

for every integer p ($a_0 = 1$, $a_1 = \frac{1}{4}$). Note that the even moments b_p, $p \geqslant 0$, of the semicircle distribution satisfy the recurrence relation

$$b_p = \frac{2p-1}{2p+2}\, b_{p-1} = \frac{(2p)!}{2^{2p} p!\,(p+1)!}\,.$$

In particular, when $p = [tN^{2/3}]$, $t > 0$,

$$N\, b_p \leqslant C\, t^{-3/2} \tag{7}$$

for some numerical $C > 0$. Now, the recurrence formula (6) easily shows that when $p \leqslant tN^{2/3}$,

$$a_p \leqslant \left(1 + \frac{t^2}{4N^{2/3}}\right)^p b_p.$$

Hence, for $p = [tN^{2/3}]$, $t > 0$, we get from (7) that

$$N a_p \leqslant C\, t^{-3/2}\, \mathrm{e}^{ct^3}. \tag{8}$$

Therefore,

$$\mathbb{P}\big\{\lambda^N_{\max} \geqslant 1 + \varepsilon\big\} \leqslant (1+\varepsilon)^{-p} N a_p \leqslant C(1+\varepsilon)^{-p} t^{-3/2}\, \mathrm{e}^{ct^3}.$$

Optimizing in $t > 0$ yields the optimal tail inequality

$$\mathbb{P}\big\{\lambda^N_{\max} \geqslant 1 + \varepsilon\big\} \leqslant C\, \mathrm{e}^{-cN\varepsilon^{3/2}} \tag{9}$$

for every $0 < \varepsilon \leqslant 1$, $N \geqslant 1$ and numerical constants C, $c > 0$.

It should be noted that inequality (9) was obtained recently by G. Aubrun [1] using bounds over the integral operators considered in [22]. Moreover, the combinatorial techniques in the evaluation of the p-th moments of the trace developed by A. Soshnikov [19] (for sample covariance matrices, see [20]) suggest the possible extension of (8) to large classes of Wigner matrices.

It may be mentioned that concentration bounds together with rates of convergence to the semicircle law σ can be used to derive a deviation inequality of $\lambda^N_{\max}$ under the level 1. For every $\varepsilon > 0$ and $N \geqslant 1$,

$$\mathbb{P}\big\{\lambda^N_{\max} \leqslant 1 - 2\varepsilon\big\} = \mathbb{P}\bigg\{\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{\lambda^N_i \leqslant 1-2\varepsilon\}} \geqslant 1\bigg\}.$$

Asymptotics of the Hermite polynomials applied to rates of convergence have been used recently in [6] to show that, for every $N \geqslant 1$ and $0 < \varepsilon \leqslant 1$,

$$\big|m^N\big((-\infty, 1-\varepsilon]\big) - \sigma\big((-\infty, 1-\varepsilon]\big)\big| \leqslant \frac{C}{\varepsilon N}$$

where $C > 0$ is a numerical constant possibly changing from line to line below. On the other hand,

$$1 - \sigma\big((-\infty, 1-\varepsilon]\big) \leqslant C\varepsilon^{3/2}$$

for every $0 < \varepsilon \leqslant 1$. In particular thus,

$$\mathbb{P}\big\{\lambda_{\max}^N \leqslant 1 - 2\varepsilon\big\} \\ \leqslant \mathbb{P}\bigg\{\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{\lambda_i^N \leqslant 1-2\varepsilon\}} - m^N\big((-\infty, 1-\varepsilon]\big) \geqslant C\Big(\varepsilon^{3/2} - \frac{1}{\varepsilon N}\Big)\bigg\}. \quad (10)$$

Let φ be the Lipschitz piecewise linear function equal to 1 on $(-\infty, 1-2\varepsilon]$ and to 0 on $[1-\varepsilon, +\infty)$. In particular,

$$\mathbb{P}\big\{\lambda_{\max}^N \leqslant 1 - 2\varepsilon\big\} \\ \leqslant \mathbb{P}\bigg\{\frac{1}{N}\sum_{i=1}^N \varphi\big(\lambda_i^N\big) - \mathbb{E}\bigg[\frac{1}{N}\sum_{i=1}^N \varphi\big(\lambda_i^N\big)\bigg] \geqslant C\Big(\varepsilon^{3/2} - \frac{1}{\varepsilon N}\Big)\bigg\}.$$

Assume now that $\varepsilon^{3/2} \geqslant 2/(\varepsilon N)$. Since φ is Lipschitz with Lipschitz coefficient ε^{-1}, measure concentration applied to the Lipschitz map $(1/N)\sum_{i=1}^N \varphi(\lambda_i^N)$ as a function of the Gaussian entries of the random matrix M^N (see [8, 4]) yields that

$$\mathbb{P}\big\{\lambda_{\max}^N \leqslant 1 - \varepsilon\big\} \leqslant C\,\mathrm{e}^{-c\varepsilon^5 N^2} \quad (11)$$

for every ε such that $cN^{-2/5} \leqslant \varepsilon \leqslant 1$, where $C, c > 0$ are numerical. Note furthermore that, together with (1), it follows from (11) that

$$\mathbb{E}\big[\lambda_{\max}^N\big] \geqslant 1 - \frac{C}{N^{2/5}}$$

for some $C > 0$.

The $\varepsilon^{3/2}$-phenomenon put forward in the GUE example may actually be shown to be quite general in the context of eigenfunction measures. We describe in the next section similar decays for measures $f^2\,\mathrm{d}\mu$ where f is a normalized eigenfunction of a hypercontractive operator with invariant probability measure μ. In the last section, we come back to the random matrix models and apply the result to the largest eigenvalues of some classes of invariant ensembles including the Gaussian and Laguerre Unitary Ensembles.

1 Concentration of eigenfunction measures

Invariant measures of hypercontractive operators satisfy equivalently a so-called logarithmic Sobolev inequality. Moreover, the typical Gaussian tail behavior of Lipschitz functions for measures satisfying logarithmic Sobolev inequalities has been studied extensively in the recent years (cf. [13]). We briefly survey a few basic results. We adopt a general framework taken from [2, 14] to which we refer for further details.

Consider a measurable space $(E, \mathcal{E})$ equipped with a probability measure μ. We denote by $\mathrm{L}^p = \mathrm{L}^p(\mu)$, $1 \leqslant p \leqslant \infty$, the Lebesgue spaces with respect to μ, and set $\|\,.\,\|_p$ to denote the norm in L^p. Let $(P_t)_{t\geqslant 0}$ be a Markov semigroup of non-negative operators, bounded and continuous on $\mathrm{L}^2(\mu)$. We denote by $\mathcal{D}_2(\mathcal{L})$ the domain in $\mathrm{L}^2(\mu)$ of the infinitesimal generator $\mathcal{L}$ of the semigroup $(P_t)_{t\geqslant 0}$. We assume that μ is invariant and reversible with respect to $(P_t)_{t\geqslant 0}$.

The fundamental theorem of L. Gross [7] connects the hypercontractivity property of $(P_t)_{t\geqslant 0}$, or $\mathcal{L}$, to the logarithmic Sobolev inequality satisfied by the invariant measure μ. Namely, if, and only if, for some $\rho > 0$,

$$\rho \int f^2 \log f^2 \, \mathrm{d}\mu \leqslant 2 \int f(-\mathcal{L}f) \, \mathrm{d}\mu \tag{12}$$

for all functions f in the domain of $\mathcal{L}$ with $\int f^2 \, \mathrm{d}\mu = 1$, then, for all $1 < p < q < \infty$ and $t > 0$ large enough so that

$$\mathrm{e}^{2\rho t} \geqslant \frac{q-1}{p-1},$$

we have

$$\|P_t f\|_q \leqslant \|f\|_p \tag{13}$$

for every f in L^p.

It is classical (see [2]) that whenever (12) holds, then

$$\rho \int f^2 \, \mathrm{d}\mu \leqslant \int f(-\mathcal{L}f) \, \mathrm{d}\mu$$

for every mean zero function f in the domain of $\mathcal{L}$. In particular, any non-trivial eigenvalue α of $-\mathcal{L}$ satisfies $\alpha \geqslant \rho$.

Classes of measures satisfying a logarithmic Sobolev inequality (12) are described in [2, 13, 14]. Some examples will be discussed in Section 2. In particular, if $\mu(\mathrm{d}x) = \mathrm{e}^{-U} \mathrm{d}x$ on $\mathbb{R}^n$ where U is such that $U - \delta\,|x|^2/2$ is convex for some $\delta > 0$, then $\rho \geqslant \delta$. The canonical Gaussian measure on $\mathbb{R}^n$ is such that $\rho = 1$.

Concentration inequalities under a logarithmic Sobolev inequality (12) may be obtained through the Herbst argument (cf. [13]). Let us call 1-Lipschitz a function F in the domain of $\mathcal{L}$ such that

$$\Gamma(f,f) = \frac{1}{2}\,\mathcal{L}\big(f^2\big) - f\mathcal{L}f \leqslant 1$$

almost everywhere. In particular, when $\mathcal{L} = \Delta - \nabla U \cdot \nabla$ with invariant measure $\mu(\mathrm{d}x) = \mathrm{e}^{-U} \mathrm{d}x$ for some smooth potential U on $\mathbb{R}^n$, $\Gamma(f,f) = |\nabla f|^2$ so that Lipschitz simply means Lipschitz in the classical Euclidean sense. Assume more generally that Γ is a derivation in the sense that $\Gamma(\varphi(f), \varphi(f)) = \varphi'(f)^2 \Gamma(f,f)$ for every smooth $\varphi : \mathbb{R} \to \mathbb{R}$. Then, under the logarithmic Sobolev inequality (12), the Herbst argument shows that whenever F is 1-Lipschitz, for every $r \geqslant 0$,

$$\mu\{F \geqslant \int F \, d\mu + r\} \leqslant e^{-\rho r^2/2}. \tag{14}$$

When applied to the Gaussian measure of the GUE and to the Lipschitz function given by the largest eigenvalue, we obtain (1).

The following is the main result on eigenfunction measures.

Theorem 1. *Let $\mathcal{L}$ be a hypercontractive operator with hypercontractive constant $\rho > 0$. Let f be an eigenfunction of $-\mathcal{L}$ with eigenvalue $\alpha > 0$. Assume that f^2 is normalized with respect to the invariant measure μ of $\mathcal{L}$, and set $d\nu = f^2 \, d\mu$. Then, whenever A is a measurable set with $\mu(A) \leqslant e^{-2\alpha(1+a)/\rho}$ for some $a > 0$, then*

$$\nu(A) \leqslant e^{-c\alpha\rho^{-1} \min(a, a^{3/2})}$$

where $c = 2\sqrt{2}/3$ (which is not sharp).

Together with (14), we get the following corollary.

Corollary 1. *Under the hypotheses of Theorem 1, let F be a 1-Lipschitz function. Then, for every $r \geqslant 0$,*

$$\nu\{F \geqslant \int F \, d\mu + 2\sqrt{\alpha}\, \rho^{-1}(1+r)\} \leqslant e^{-c\alpha\rho^{-1} \max(r^2, r^{3/2})}.$$

Proof of Theorem 1. By Hölder's inequality, for every $r > 1$,

$$\nu(A) = \int_A f^2 \, d\mu \leqslant \mu(A)^{1-(1/r)} \|f\|_{2r}^2.$$

Since $P_t f = e^{-\alpha t} f$, hypercontractivity (13) shows that

$$\|f\|_{2r} \leqslant (2r-1)^{\alpha/2\rho}.$$

Hence

$$\nu(A) \leqslant e^{-\alpha\rho^{-1}[2(1+a)(1-1/r) - \log(2r-1)]}.$$

Optimizing over $r > 1$ yields that

$$\nu(A) \leqslant e^{-2\alpha\rho^{-1} \varphi(a)}$$

where

$$\varphi(a) = \sqrt{a}\,\sqrt{1+a} - \log\left(\frac{\sqrt{1+a} + \sqrt{a}}{\sqrt{1+a} - \sqrt{a}}\right).$$

Noticing that the derivative of $\varphi(a^2)$ is equal to

$$\frac{2a^2}{\sqrt{1+a^2}} \geqslant \sqrt{2} \min(a, a^2),$$

the conclusion easily follows. The proof is complete. $\square$

2 Application to the largest eigenvalues of random matrices

Before turning to the application sketched in the introduction, it might be worthwhile mentioning the following general observation of possible independent interest in the description of the eigenvalue distribution. Coming back to the distribution (2) of the eigenvalues $(\lambda_1^N, \dots, \lambda_N^N)$ of the GUE, the Vandermonde determinant

$$H_N(x) = \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j), \qquad x = (x_1, \dots, x_N) \in \mathbb{R}^N,$$

is actually an eigenvector of the Ornstein–Uhlenbeck generator $\mathcal{L} = \Delta - x \cdot \nabla$ in $\mathbb{R}^N$, with eigenvalue $N(N-1)/2$. Denote by γ_N the canonical Gaussian measure on $\mathbb{R}^N$. As a consequence of Theorem 1, we thus get the following result that describes bounds on the distribution of the eigenvalues in terms of the corresponding Gaussian measure.

Corollary 2. *Let A be a Borel set in $\mathbb{R}^N$ such that $\gamma_N(A) \leqslant \mathrm{e}^{-N(N-1)(1+a)}$ for some $a > 0$. Then,*

$$\mathbb{P}\big\{ \big(\lambda_1^N, \dots, \lambda_N^N\big) \in A \big\} \leqslant \mathrm{e}^{-cN(N-1)\min(a, a^{3/2})}$$

where $c > 0$ is numerical.

Together with the concentration inequality (14) for γ_N, it follows that whenever $F : \mathbb{R}^N \to \mathbb{R}$ is 1-Lipschitz,

$$\mathbb{P}\Big\{ F\big(\lambda_1^N, \dots, \lambda_N^N\big) \geqslant \int F(x)\, \gamma_N(\mathrm{d}x) + \sqrt{2N(N-1)}\,(1+r) \Big\} \leqslant \mathrm{e}^{-cN(N-1)\max(r^2, r^{3/2})}$$

for every $r \geqslant 0$.

Examples where the setting of Section 1 applies are as follows. Let I be some interval of the real line, and let μ be a probability measure on the Borel sets of I such that $\int \mathrm{e}^{c|x|}\, \mu(\mathrm{d}x) < \infty$ for some $c > 0$. Denote by $(Q_k)_{k \in \mathbb{N}}$ the orthonormal polynomials of the probability measure μ. We assume that there exists a Markov semigroup $(P_t)_{t \geqslant 0}$ with invariant measure μ such that the spectral decomposition of the generator $\mathcal{L}$ of $(P_t)_{t \geqslant 0}$ is actually given by the polynomials Q_k in the sense that there exist $\alpha_k \geqslant 0$, $k \in \mathbb{N}$, such that for each k and t,

$$P_t Q_k = \mathrm{e}^{-\alpha_k t} Q_k.$$

In other words, $\mathcal{L} Q_k = -\alpha_k Q_k$, $k \in \mathbb{N}$. See e.g. [16].

The classical orthogonal polynomials (cf. [21]) are well-known to enter this setting. Let us mention the Hermite polynomials $(h_k)_{k \in \mathbb{N}}$ orthonormal with respect to the canonical Gaussian measure $\gamma(\mathrm{d}x) = \mathrm{e}^{-x^2/2}\, \mathrm{d}x / \sqrt{2\pi}$ on

$I = \mathbb{R}$. The Hermite polynomials h_k, $k \in \mathbb{N}$, are eigenfunctions of the Ornstein-Uhlenbeck operator $\mathcal{L}f = f'' - xf'$ with respective eigenvalues $-k$. As we have seen, γ satisfies the logarithmic Sobolev inequality (12) with $\rho = 1$ and for every smooth enough function f, $\Gamma(f,f) = {f'}^2$. Similarly for the Laguerre polynomials $(L_k^\theta)_{k\in\mathbb{N}}$, $\theta > -1$, orthonormal with respect to $\mu^\theta(\mathrm{d}x) = \Gamma(\theta+1)^{-1}x^\theta \mathrm{e}^{-x}\,\mathrm{d}x$ on $I = (0,\infty)$, and associated with the Laguerre operator $\mathcal{L}^\theta f = xf'' + (\theta + 1 - x)f'$. For every $k \in \mathbb{N}$, $\mathcal{L}^\theta L_k^\theta = -kL_k^\theta$. In this example, $\Gamma(f,f) = x{f'}^2$ and the logarithmic Sobolev constant of μ^θ may be shown to be equal to $1/2$, at least for $\theta \geqslant -1/2$ (cf. [2]). On $I = (-1,+1)$, we may consider more generally the Jacobi polynomials $(J_k^{a,b})_{k\in\mathbb{N}}$, $a, b > -1$, orthonormal with respect to $\mu^{a,b}(\mathrm{d}x) = C_{a,b}(1+x)^a(1-x)^b\,\mathrm{d}x$. They are eigenfunctions of the Jacobi operator

$$\mathcal{L}^{a,b}f = \big(1-x^2\big)f'' + \big(a - b - (a+b+2)x\big)f'$$

with eigenvalues $-k(k+a+b+1)$, $k \in \mathbb{N}$. We have here $\Gamma(f,f) = (1-x^2){f'}^2$ while, when $a = b$, $\rho = 2(a+1)$ (cf. [2]).

If M^N is a matrix from the GUE, its entries consist of random variables M_{ij}^N, $1 \leqslant i,j \leqslant N$ such that M_{ij}^N, $i \leqslant j$, are independent complex (real when $i = j$) centered Gaussian variables with variances $1/(4N)$. The mean spectral measure is given by (3). Since the Gaussian measure γ has hypercontractivity constant 1, it follows from Theorem 1, or rather the developments of the introduction, that

$$\mathbb{P}\big\{\lambda_{\max}^N \geqslant 1 + \varepsilon\big\} \leqslant C\min\big(1, \sqrt{\varepsilon}\big)^{-1}\mathrm{e}^{-cN\max(\varepsilon^2,\varepsilon^{3/2})}$$

for numerical constants C, $c > 0$ and all $\varepsilon > 0$, $N \geqslant 1$.

Let now $M^N = M = B^*B$ where B is an $N \times N$ random matrix whose entries consist of independent complex centered Gaussian variables with variances $1/(4N)$. The mean spectral measure m^N of M^N converges as $N \to \infty$ to the image of the semicircle law under the map $x \mapsto x^2$ and the largest eigenvalue converges almost surely to the right-hand side of the support. See [9] for a discussion where it is shown in particular that for every bounded measurable function $f : (0,\infty) \to \mathbb{R}$,

$$\int_{\mathbb{R}} f\,\mathrm{d}m^N = \int_0^\infty f\Big(\frac{x}{4N}\Big)\frac{1}{N}\sum_{k=0}^{N-1}\big(L_k^0\big)^2(x)\,\mu^0(\mathrm{d}x)$$

where we recall that $(L_k^0)_{k\in\mathbb{N}}$ are the Laguerre polynomials of parameter $\theta = 0$. The Laguerre operator $\mathcal{L}^0$ is hypercontractive with constant $1/2$. We then get as before that

$$\mathbb{P}\big\{\lambda_{\max}^N \geqslant 1 + \varepsilon\big\} \leqslant C\min\big(1, \sqrt{\varepsilon}\big)^{-1}\mathrm{e}^{-cN\min(\varepsilon,\varepsilon^{3/2})}$$

for numerical constants C, $c > 0$ and all $\varepsilon > 0$, $N \geqslant 1$. Asymptotically, the result applies similarly to products B^*B of rectangular $N \times K$ matrices provided that $K/N \to 1$ as $N \to \infty$.

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