Estimates of the Solutions of a System of Quasi-linear PDEs. A Probabilistic Scheme.

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Summary. We propose a probabilistic scheme to estimate the Hölder norm and the gradient of the solutions of a system of quasi-linear PDEs of parabolic type. Indeed, thanks to the theory of Forward Backward stochastic differential equations, we are able to give a stochastic representation of the solutions of such systems of PDEs. Making use of Krylov and Safonov estimates, we deduce a Hölder estimate of these solutions in the case of uniformly parabolic systems with measurable coefficients. Moreover, from a variant of the Malliavin–Bismut integration by parts formula, we establish under appropriate assumptions an estimate of the supremum norm of the gradient of these solutions.

Résumé. Nous proposons une démarche probabiliste pour estimer la norme Hölder ainsi que le gradient des solutions d'un système d'EDPs quasi-linéaires de type parabolique. En effet, à l'aide de la théorie des équations différentielles stochastiques progressives rétrogrades, nous sommes capables de donner une représentation stochastique des solutions de tels systèmes d'EDPs. En appliquant les estimations de Krylov et Safonov, nous déduisons une estimation Hölder de ces solutions dans le cas de systèmes uniformément paraboliques à coefficients mesurables. De plus, à l'aide d'une variante de la formule d'intégration par parties de Malliavin–Bismut, nous établissons sous des hypothèses appropriées une estimation de la norme supremum du gradient de ces solutions.

Key words: Forward-backward stochastic differential equation, gradient estimate, Hölder estimate, integration by parts, system of quasi-linear PDEs of parabolic type.

Introduction

Let us firstly recall that in our paper Delarue [6], we establish a theorem of existence and uniqueness of solutions to Forward-Backward SDEs in the case of a non-degenerate diffusion matrix. Basically, this theorem is proved in two steps. First, applying a fixed point theorem, we obtain by means of purely probabilistic tools a unique solvability result in the case of a small enough time duration. Then, in a second part, using a gradient estimate of the solutions

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of a system of quasilinear PDEs of parabolic type, given in the monograph of Ladyzhenskaya et al. [18], we deduce thanks to the non-degeneracy assumption a global existence and uniqueness result.

The purpose of this paper is simply to develop a probabilistic scheme to establish such an estimate.

Of course, several articles have already proposed some probabilistic approaches to establish estimates of the solutions of a second order PDE. Among them, the Krylov and Safonov estimate, proved in Krylov and Safonov [15], is certainly one of the most famous results. Indeed, this fundamental work has permitted to prove the Hölder continuity of the solutions of a linear second order PDE of nondivergence type with measurable coefficients, and then to extend to such operators the older result due to De Giorgi and Nash related to the divergent case.

Moreover, to obtain from a probabilistic point of view a gradient estimate of the solutions of a second order PDE, the now well-known theory of stochastic flows plays an essential role. Actually, let us assume that for $(t,x) \in [0,T] \times \mathbb{R}^P$, $X^{t,x}$ stands for the solution, starting from x at time t, of a stochastic differential equation associated to a differential operator $\mathcal{L}$, then, it is well known that such a theory permits to study the regularity of the process X upon the parameter (t,x), and therefore, thanks to the Feynman–Kac formula, very successfully investigate the regularity of the solutions of a PDE associated to the operator $\mathcal{L}$ and defined on the whole set $[0,T] \times \mathbb{R}^P$.

However, this approach may be fruitless in many cases. Indeed, assume for example that $\mathcal{D}$ is a cylinder of the form $[0,T[\times \{x \in \mathbb{R}^P,\ |x| < R\}$ and that u is a harmonic function on $\mathcal{D}$ with respect to $\mathcal{L}$ and admits from the Feynman–Kac formula the following representation:

$$\forall\,(t,x) \in \overline{\mathcal{D}}, \qquad u(t,x) = \mathbb{E}\big[u\big(\tau^{t,x}, X^{t,x}_{\tau^{t,x}}\big)\big], \tag{0.1}$$

where for every $(t,x) \in \overline{\mathcal{D}}$, $\tau^{t,x}$ stands for the first exit time of $(s, X_s^{t,x})_{t\leqslant s\leqslant T}$ from $\mathcal{D}$. Then, estimating the gradient of the function u by differentiating the expression (0.1) is unfortunately hopeless since the function $(t,x) \in \mathcal{D} \mapsto \tau^{t,x}$ may be not differentiable with respect to x.

Actually, several articles have proposed some schemes to go around such an obstacle. For example, Krylov [14] has developed the notion of quasiderivatives of the solution of a stochastic equation. Inspired by this work and by the earlier papers of Bismut [2] and Elworthy and Li [7], Thalmaier [26] has proposed a variant of the Malliavin–Bismut integration by parts formula, which has been successfully applied in Thalmaier and Wang [27] to establish a gradient estimate of interior type of the solutions of a linear elliptic equation in the more general framework of manifolds.

Let us also mention that, in another direction but still in the framework of manifolds, Cranston ([4] and [5]) and Wang ([28] and [29]), have proposed to estimate the gradient of a harmonic function by using earlier techniques of coupling of two Brownian motions.

In our paper, using the theory of Forward-Backward SDEs to represent the solutions of a system of quasi-linear PDEs (see the papers of Ma et al. [20], Pardoux and Tang [24] and Delarue [6] on this subject), we successfully adapt to our case the Krylov and Safonov result and the Thalmaier approach. Actually, we obtain in a first step interior estimates of Hölder type of the solutions of a uniformly parabolic system of quasilinear PDEs with measurable coefficients (see also Ladyzhenskaya and Ural'tseva [19] for an analytical point of view). Under certain assumptions on the initial condition, we deduce global Hölder estimates of these solutions. In a second step, we establish under appropriate assumptions on the coefficients both interior and global estimates of the supremum norm of the gradient of these solutions, and in particular, we deduce the gradient estimate that we used in our previous paper Delarue [6].

Moreover, let us mention that we also show, as a side result, how we can deduce from the scheme given in Thalmaier [26] some properties of differentiability of the function u given by (0.1).

Hence, the paper is organized as follows: section 1 is devoted to the study of Hölder regularity of the solutions, and we deduce in section 2 estimates of the gradient of these solutions.

Frequently Used Notations

- *$\forall N \in \mathbb{N}^*$, $\langle .\,,.\rangle$ and $|.|$ denote the Euclidean scalar product and the Euclidean norm on $\mathbb{R}^N$.*
- *$\forall N \in \mathbb{N}^*$, $\forall x \in \mathbb{R}^N$, $\forall i \in \{1,\dots,N\}$, x_i denotes the i^{th} coordinate of the vector x.*
- *$\forall N \in \mathbb{N}^*$, $\forall M \in \mathbb{N}^*$, $\forall x \in \mathbb{R}^{M\times N}$, $\forall i \in \{1,\dots,M\}$, x_i denotes the i^{th} row of the matrix x.*
- *$\forall N \in \mathbb{N}^*$, $\forall M \in \mathbb{N}^*$, $\forall x \in \mathbb{R}^{M\times N}$, x^T denotes the transposed of the matrix x.*
- *$\forall N \in \mathbb{N}^*$, $\forall x \in \mathbb{R}^N$ and $\forall R \geqslant 0$, $B_N(x,R)$ and $\overline{B}_N(x,R)$ denote the open Euclidean ball of dimension N, of center x and of radius R, and the closed Euclidean ball of dimension N, of center x and of radius R.*
- *$\forall N \in \mathbb{N}^*$, μ_N denotes the Lebesgue measure on $\mathbb{R}^N$.*
- *The notation $[.\,,.]$ stands for the quadratic covariation bracket.*

System of Quasi-linear PDEs and FBSDEs

We now introduce the system of quasi-linear PDEs that will be studied in this paper, and then recall the principle of the probabilistic representation. As a first consequence, we give an L^∞-estimate of the solutions that will be crucial in the sequel of the article.

Let T be a positive real, and

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^{Q\times P} \longrightarrow \mathbb{R}^P \\ f &: [0,T] \times \mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^{Q\times P} \longrightarrow \mathbb{R}^Q \\ \sigma &: [0,T] \times \mathbb{R}^P \times \mathbb{R}^Q \longrightarrow \mathbb{R}^{P\times P} \\ H &: \mathbb{R}^P \longrightarrow \mathbb{R}^Q \end{aligned} \qquad \textbf{(A.0)}$$

be measurable functions with respect to the Borel σ-fields.

Assumption (A). We say that the functions b, f, H and σ satisfy Assumption (A) if there exist four constants $\alpha_0 > 0$, L, $\lambda > 0$ and Λ, such that they satisfy (A.0) as well as the following properties:

(A.1) $\forall\, t \in [0,T]$, $\forall\,(x,y,z) \in \mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^{Q\times P}$,

$$\begin{aligned} |b(t,x,y,z)| &\leqslant \Lambda \times \big(1 + |y| + |z|\big), \\ |f(t,x,y,z)| &\leqslant \Lambda \times \big(1 + |y| + |z|\big), \\ |\sigma(t,x,y)| &\leqslant \Lambda \times \big(1 + |y|\big), \\ |H(x)| &\leqslant \Lambda. \end{aligned}$$

(A.2) $\forall\,(t,x,y) \in [0,T] \times \mathbb{R}^P \times \mathbb{R}^Q$,

$$\forall\,\zeta \in \mathbb{R}^P, \qquad \langle \zeta, a(t,x,y)\zeta\rangle \geqslant \lambda\,|\zeta|^2,$$

where the function a is defined as follows on $[0,T] \times \mathbb{R}^P \times \mathbb{R}^Q$:

$$\forall\,(t,x,y) \in [0,T] \times \mathbb{R}^P \times \mathbb{R}^Q, \qquad a(t,x,y) = \sigma\sigma^*(t,x,y).$$

(A.3) $\forall\,(x,x') \in (\mathbb{R}^P)^2$, $|H(x') - H(x)| \leqslant L\,|x' - x|^{\alpha_0}$.

Notations. Let

$$W^{1,2,P+1}_{\mathrm{loc}}\big(]0,T[\, \times \mathbb{R}^P, \mathbb{R}^Q\big)$$

be the set of all functions $u :]0,T[\, \times \mathbb{R}^P \longrightarrow \mathbb{R}^Q$ such that, for all $R > 0$,

$$\int_{]0,T[\times B_P(0,R)} \big(|u|^{P+1} + |u'_t|^{P+1} + |u'_x|^{P+1} + |u''_{x,x}|^{P+1}\big)\, \mathrm{d}\mu_{P+1} < \infty. \qquad (0.2)$$

We recall from Lemma 3.3 of Chapter II of Ladyzhenskaya et al. [18] that $W^{1,2,P+1}_{\mathrm{loc}}(]0,T[\, \times \mathbb{R}^P, \mathbb{R}^Q)$ is embedded in $C([0,T] \times \mathbb{R}^P, \mathbb{R}^Q)$, i.e., for each function $u \in W^{1,2,P+1}_{\mathrm{loc}}(]0,T[\, \times \mathbb{R}^P, \mathbb{R}^Q)$, there exists a continuous function on $[0,T] \times \mathbb{R}^P$ which is equal to u almost everywhere. We will always be considering this function.

Under these notations, we assume that there exists

$$\theta \in W^{1,2,P+1}_{\mathrm{loc}}\big(]0,T[\, \times \mathbb{R}^P, \mathbb{R}^Q\big) \cap L^\infty\big([0,T] \times \mathbb{R}^P, \mathbb{R}^Q\big),$$

solution of the following system:

$$(\mathcal{E})\qquad \begin{cases} \forall\,(t,x)\in[0,T]\times\mathbb{R}^P,\ \forall\,\ell\in\{1,\dots,Q\},\\ \dfrac{\partial\theta_\ell}{\partial t}(t,x)+\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{P} a_{i,j}(t,x,\theta(t,x))\dfrac{\partial^2\theta_\ell}{\partial x_i\partial x_j}(t,x)\\ \qquad +\displaystyle\sum_{i=1}^{P} b_i\big(t,x,\theta(t,x),\nabla_x\theta(t,x)\,\sigma\big(t,x,\theta(t,x)\big)\big)\dfrac{\partial\theta_\ell}{\partial x_i}(t,x)\\ \qquad + f_\ell(t,x,\theta(t,x),\nabla_x\theta(t,x)\,\sigma(t,x,\theta(t,x))\big)=0,\\ \forall\,x\in\mathbb{R}^P,\ \theta(T,x)=H(x). \end{cases}$$

Representation of the function θ. Thanks to the theory of FBSDEs, we firstly give a stochastic representation of the function θ.

To this aim, we consider $(t,x)\in[0,T[\times\mathbb{R}^P$. Thanks to Assumption (A) and thanks to Theorem 1, Paragraph 6, Chapter II of Krylov [13], we know that there exists a triple $((X,B),(\Omega,\mathcal{F},\mathbb{P}),(\mathcal{F}_s)_{t\leqslant s\leqslant T})$, which is a weak solution of the SDE:

$$\forall\,s\in[t,T],\qquad X_s=x+\int_t^s\sigma\big(r,X_r,\theta(r,X_r)\big)\,\mathrm{d}B_r. \tag{0.3}$$

Hence, we can define:

$$\forall\,s\in[t,T],\qquad Y_s=\theta(s,X_s),\qquad Z_s=\nabla_x\theta(s,X_s)\,\sigma(s,X_s,Y_s). \tag{0.4}$$

Thanks to Theorem 4, Paragraph 2, Chapter II of Krylov [13], note that the process Z is correctly defined up to a $\mu_1\otimes\mathbb{P}$ neglictible set, i.e., if $\widetilde{\nabla_x\theta}$ coincides with $\nabla_x\theta$ almost everywhere, then the associated process $\widetilde{Z}$ is equal to Z up to a $\mu_1\otimes\mathbb{P}$ neglictible set.

The following proposition details the link between FBSDEs and PDEs (see also Ma et al. [20], Pardoux and Tang [24] or Delarue [6] on this point):

Proposition 0.1. *Let τ be an $(\mathcal{F}_s)_{t\leqslant s\leqslant T}$ stopping time such that:*

$$\exists\,m\geqslant 0,\qquad \mathbb{P}\Big\{\sup_{t\leqslant s\leqslant\tau}|X_s|\leqslant m\Big\}=1. \tag{0.5}$$

Then, the process $(X_s,Y_s,Z_s)_{t\leqslant s\leqslant\tau}$ defined in (0.4) *satisfies the following FBSDE:*

$$\begin{cases} X_s=x+\displaystyle\int_t^s\sigma(r,X_r,Y_r)\,\mathrm{d}B_r,\\ Y_s=\theta(\tau,X_\tau)+\displaystyle\int_s^\tau e(r,X_r,Y_r,Z_r)\,\mathrm{d}r-\int_s^\tau Z_r\,\mathrm{d}B_r,\\ \textit{for all } s\in[t,\tau], \textit{ and } \mathbb{E}\Big[\displaystyle\int_t^\tau\big(|X_s|^2+|Y_s|^2+|Z_s|^2\big)\,\mathrm{d}s\Big]<\infty, \end{cases} \tag{0.6}$$

where, for all $(s,u,v,w) \in [0,T] \times \mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^{Q\times P}$,

$$e(s,u,v,w) = w\big(\sigma^{-1}b\big)(s,u,v,w) + f(s,u,v,w). \tag{0.7}$$

Proof. From the Sobolev embedding theorems, note that there exists a constant $C^{(0.1)}$, only depending on m, P, Q and T, such that:

$$\mu_{P+1}\Bigg\{(r,y) \in [0,T] \times \overline{B}_P(0,m), |\nabla_x\theta(r,y)| > C^{(0.1)} \times \left(\int_{B_P(0,m)} \big(|\theta'_x| + |\theta''_{x,x}|\big)^{P+1}(r,u)\,\mathrm{d}u\right)^{1/(P+1)}\Bigg\} = 0. \tag{0.8}$$

Hence, almost surely:

$$\begin{aligned}
&\int_t^T |\nabla_x\theta(r,X_r)|^2\,\mathrm{d}r \\
&\quad \leqslant C^{(0.1)} \int_t^T \left(\int_{B_P(0,m)} \big(|\theta'_x| + |\theta''_{x,x}|\big)^{P+1}(r,u)\,\mathrm{d}u\right)^{\frac{2}{P+1}} \mathrm{d}r \qquad (0.9)\\
&\quad \leqslant C^{(0.1)}(T-t)^{\frac{P-1}{P+1}} \left(\int_t^T \int_{B_P(0,m)} \big(|\theta'_x| + |\theta''_{x,x}|\big)^{P+1}(r,u)\,\mathrm{d}u\,\mathrm{d}r\right)^{\frac{2}{P+1}}.
\end{aligned}$$

Hence, thanks to the system of PDEs $(\mathcal{E})$ and thanks to Theorem 1, Paragraph 10, Chapter II of Krylov [13], we deduce (0.6). □

L^∞**-estimate of** θ**.** Actually, the representation (0.6) is the main tool that we will employ to estimate the function θ. As a first application of this deep connection between FBSDEs and PDEs, we give the following bound of $\|\theta\|_\infty$:

Theorem 0.1. *Under Assumption (A), there exists a constant* M_0 *only depending on* Λ *and* T*, such that the following estimate holds:*

$$\forall\,(t,x) \in [0,T] \times \mathbb{R}^P, \qquad |\theta(t,x)| \leqslant M_0. \tag{0.10}$$

Proof. Consider $(t,x) \in [0,T] \times \mathbb{R}^P$. Keeping the notations (0.3) and (0.4), we define for every $n \in \mathbb{N}$:

$$t^n = \inf\{t \leqslant s \leqslant T,\ |X_s - x| \geqslant n\}, \qquad \text{where } \inf\varnothing = T, \tag{0.11}$$

as well as the following process:

$$\forall\, s \in [t,T], \qquad B_s^n = B_s - \int_t^{s\wedge t^n} \big(\sigma^{-1}b\big)(r,X_r,Y_r,Z_r)\,\mathrm{d}r. \tag{0.12}$$

Then, from (0.9) and the Novikov condition, we know from the Girsanov theorem that B^n is an $(\mathcal{F}_s)_{t\leqslant s\leqslant T}$-Brownian motion under the probability measure $\mathbb{P}^n$ given by:

$$\forall\, t \leqslant s \leqslant T, \qquad \frac{\mathrm{d}\mathbb{P}^n}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_s} = \exp\biggl(\int_t^{s\wedge t^n} \bigl\langle \bigl(\sigma^{-1}b\bigr)(r, X_r, Y_r, Z_r), \mathrm{d}B_r\bigr\rangle - \frac{1}{2}\int_t^{s\wedge t^n} \bigl|\bigl(\sigma^{-1}b\bigr)(r, X_r, Y_r, Z_r)\bigr|^2 \,\mathrm{d}r\biggr). \quad (0.13)$$

The expectation under $\mathbb{P}^n$ is denoted by $\mathbb{E}^n$.

Then, thanks to Proposition 0.1 and to (0.9), we deduce that the process $(X_s, Y_s, Z_s)_{t\leqslant s\leqslant t^n}$ satisfies the FBSDE:

$$\begin{cases} X_s = x + \displaystyle\int_t^s b(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}B_r^n, \\ Y_s = \theta(t^n, X_{t^n}) + \displaystyle\int_s^{t^n} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t^n} Z_r\,\mathrm{d}B_r^n, \\ \text{for all } s \in [t, t^n], \text{ and } \mathbb{E}^n\biggl[\displaystyle\int_t^{t^n} \bigl(|X_s|^2 + |Y_s|^2 + |Z_s|^2\bigr)\,\mathrm{d}s\biggr] < \infty. \end{cases} \quad (0.14)$$

Referring to Pardoux [22], we deduce that there exists a constant $C^{(0.2)}$, only depending on Λ and T, such that for every $n \in \mathbb{N}$:

$$\mathbb{E}^n\Bigl[\sup_{t\leqslant s\leqslant t^n} |Y_s|^2\Bigr] + \mathbb{E}^n\biggl[\int_t^{t^n} |Z_s|^2\,\mathrm{d}s\biggr] \leqslant C^{(0.2)}\bigl(1 + \mathbb{E}^n\bigl[|\theta(t^n, X_{t^n})|^2\bigr]\bigr). \quad (0.15)$$

Therefore, for every $n \in \mathbb{N}$:

$$\begin{aligned} &\mathbb{E}^n\Bigl[\sup_{t\leqslant s\leqslant t^n} |Y_s|^2\Bigr] + \mathbb{E}^n\biggl[\int_t^{t^n} |Z_s|^2\,\mathrm{d}s\biggr] \\ &\quad\leqslant C^{(0.2)}\bigl(1 + \mathbb{E}^n\bigl[|H(X_T)|^2\bigr] + \|\theta\|_\infty^2\,\mathbb{P}^n\{t^n < T\}\bigr) \\ &\quad\leqslant C^{(0.2)}\biggl(1 + \mathbb{E}^n\bigl[|H(X_T)|^2\bigr] + \frac{1}{n^2}\,\|\theta\|_\infty^2\,\mathbb{E}^n\Bigl[\sup_{t\leqslant s\leqslant T} |X_s - x|^2\Bigr]\biggr). \end{aligned} \quad (0.16)$$

Moreover, noting that $(X_s)_{t\leqslant s\leqslant T}$ satisfies the following equation:

$$X_s = x + \int_t^{s\wedge t^n} b(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}B_r^n, \quad (0.17)$$

for all $s \in [t, T]$, we deduce from (0.15) that there exists a constant $C^{(0.3)}$, only depending on Λ and T, such that for every $n \in \mathbb{N}$:

$$\mathbb{E}^n\Bigl[\sup_{t\leqslant s\leqslant T} |X_s - x|^2\Bigr] \leqslant C^{(0.3)}\bigl(1 + \|\theta\|_\infty^2\bigr). \quad (0.18)$$

Therefore, injecting (0.18) in (0.16), and letting $n \to +\infty$, we complete the proof. □

1 Hölder Estimate of the Solution

In this section, we assume that the coefficients satisfy Assumption (A).

The goal of this section is to prove, under Assumption (A), a Hölder estimate of the function θ. To reach such an aim, we make use of the Krylov and Safonov estimate (see Krylov and Safonov [15]), and, thanks to the representation formula (0.6), we adapt their scheme to the quasilinear case.

Note that Theorem 0.1 plays an essential role in the whole section.

We firstly introduce the following notations:

Notations. For all $(t,x) \in [0,T] \times \mathbb{R}^P$ and all $0 \leqslant R \leqslant (T-t)^{1/2}$, let:

$$\mathcal{Q}_{(t,x)}(R) = \big\{(s,y) \in [0,T] \times \mathbb{R}^P, \\ 0 \leqslant s-t \leqslant R^2,\ \max_{i=1,\dots,P} |y_i - x_i| \leqslant R\big\}. \quad (1.1)$$

Here is the main result of this section:

Theorem 1.1. *There exist two constants $\Gamma^{(1.1)}$ and $\alpha > 0$, only depending on λ, Λ, P, Q and T, such that for all $(t,x) \in [0,T[\times\mathbb{R}^P$, $0 < R \leqslant (T-t)^{1/2}$, and $i \in \{1,\dots,Q\}$,*

$$\operatorname*{osc}_{\mathcal{Q}_{(t,x)}(R)}(\theta_i) \leqslant \Gamma^{(1.1)}\left(\left(\frac{R}{R_0(t)}\right)^{\alpha} w_0(t,x) + RR_0(t)\right), \quad (1.2)$$

where:

$$\begin{cases} w_0(t,x) = \displaystyle\max_{i=1,\dots,Q,\ \varepsilon=\pm1}\Big(\operatorname*{osc}_{\mathcal{Q}_{(t,x)}(R_0(t))}\big(10\,\varepsilon\, QM_0\theta_i + |\theta|^2\big)\Big), \\ R_0(t) = (T-t)^{1/2}. \end{cases} \quad (1.3)$$

Proof. Fix $(t,x) \in [0,T[\times\mathbb{R}^P$. Following the proof of Proposition 0.1, consider a weak solution $((X,B),(\Omega,\mathcal{F},\mathbb{P}),(\mathcal{F}_s)_{t\leqslant s\leqslant T})$ of the SDE:

$$X_s = x + \int_t^s \sigma\big(r, X_r, \theta(r,X_r)\big)\,\mathrm{d}B_r. \quad (1.4)$$

Consider an $(\mathcal{F}_s)_{t\leqslant s\leqslant T}$ stopping time τ, such that:

$$\exists\, m \geqslant 0, \qquad \mathbb{P}\Big\{\sup_{t\leqslant s\leqslant\tau} |X_s| \leqslant m\Big\} = 1. \quad (1.5)$$

Hence, we know that the process $(X_s,Y_s,Z_s)_{t\leqslant s\leqslant\tau}$ defined in (0.4) satisfies the following FBSDE:

$$\begin{cases} X_s = x + \displaystyle\int_t^s \sigma(r,X_r,Y_r)\,\mathrm{d}B_r, \\ Y_s = \theta(\tau,X_\tau) + \displaystyle\int_s^\tau e(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_s^\tau Z_r\,\mathrm{d}B_r, \\ \text{for all } s \in [t,\tau], \text{ and } \mathbb{E}\Big[\displaystyle\int_t^\tau \big(|X_s|^2 + |Y_s|^2 + |Z_s|^2\big)\,\mathrm{d}s\Big] < \infty. \end{cases} \quad (1.6)$$

Hence, for all $i \in \{1, \dots, Q\}$ and $\mu \in \mathbb{R}$, we have for every $s \in [t, \tau]$:

$$\begin{aligned}\mu (Y_s)_i + |Y_s|^2 = \mu\, \theta_i(\tau, X_\tau) + |\theta(\tau, X_\tau)|^2 \\ + \int_s^\tau \big(\mu f_i + 2\langle Y_r, f\rangle\big)(r, X_r, Y_r, Z_r)\,\mathrm{d}r \\ + \int_s^\tau \big(\mu (Z_r)_i + 2\, Y_r^* Z_r\big)\big(\sigma^{-1} b\big)(r, X_r, Y_r, Z_r)\,\mathrm{d}r \\ - \int_s^\tau |Z_r|^2\,\mathrm{d}r - \int_s^\tau \big(\mu (Z_r)_i + 2\, Y_r^* Z_r\big)\,\mathrm{d}B_r.\end{aligned} \tag{1.7}$$

Fix $(i, \mu) \in \{1, \dots, Q\} \times \mathbb{R}$ and put:

$$\forall\, r \in [t, T], \qquad Z_r^+ = \mu (Z_r)_i + 2\, Y_r^* Z_r. \tag{1.8}$$

Then, we have for every $s \in [t, \tau]$:

$$\begin{aligned}\mu (Y_s)_i + |Y_s|^2 = \mu\, \theta_i(\tau, X_\tau) + |\theta(\tau, X_\tau)|^2 \\ + \int_s^\tau \big(\mu f_i + 2\langle Y_r, f\rangle\big)(r, X_r, Y_r, Z_r)\,\mathrm{d}r \\ + \int_s^\tau Z_r^+ \big(\sigma^{-1} b\big)(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^\tau |Z_r|^2\,\mathrm{d}r \\ - \int_s^\tau Z_r^+\,\mathrm{d}B_r.\end{aligned} \tag{1.9}$$

In particular, thanks to Theorem 0.1, we can find $C^{(1.1)}$, only depending on Λ, λ and T, such that for all $t \leqslant s \leqslant s' \leqslant \tau$:

$$\begin{aligned}\mu (Y_s)_i + |Y_s|^2 \leqslant \mu (Y_{s'})_i + |Y_{s'}|^2 + \big(1 + \mu^2\big)\, C^{(1.1)} \int_s^{s'} \mathrm{d}r \\ + C^{(1.1)} \int_s^{s'} |Z_r^+|^2\,\mathrm{d}r - \int_s^{s'} Z_r^+\,\mathrm{d}B_r.\end{aligned} \tag{1.10}$$

From Kobylanski [12], we know that there exists a unique progressively measurable process, denoted by $(\overline{Y}_s, \overline{Z}_s)_{t \leqslant s \leqslant \tau}$, satisfying:

$$\begin{cases} \exists\, c^{(1.1)} \geqslant 0, \quad \mathbb{P}\big\{\forall\, t \leqslant s \leqslant \tau,\ \big|\overline{Y}_s\big| \leqslant c^{(1.1)}\big\} = 1, \\ \mathbb{E}\Big[\displaystyle\int_t^\tau \big|\overline{Z}_s\big|^2\,\mathrm{d}s\Big] < \infty, \end{cases} \tag{1.11}$$

as well as the following BSDE:

$$\begin{aligned}\overline{Y}_s = \mu\, \theta_i(\tau, X_\tau) + |\theta(\tau, X_\tau)|^2 \\ + (1 + \mu^2)\, C^{(1.1)} \int_s^\tau \mathrm{d}r + C^{(1.1)} \int_s^\tau \big|\overline{Z}_r\big|^2\,\mathrm{d}r - \int_s^\tau \overline{Z}_r\,\mathrm{d}B_r.\end{aligned} \tag{1.12}$$

From the comparaison principle stated in Kobylanski [12], we deduce that:

$$\mu\,\theta_i(t,x) + |\theta(t,x)|^2 \leqslant \overline{Y}_t. \tag{1.13}$$

Let us now prove the following lemma:

Lemma 1.1. *The following Novikov condition is satisfied:*

$$\mathbb{E}\left[\exp\left(\frac{\left(C^{(1.1)}\right)^2}{2}\int_t^\tau \left|\overline{Z}_r\right|^2 \mathrm{d}r\right)\right] < +\infty. \tag{1.14}$$

Proof. Let us define for every $n \in \mathbb{N}^*$:

$$\tau_n = \inf\left\{t \leqslant s \leqslant \tau,\ \int_t^s \left|\overline{Z}_r\right|^2 \mathrm{d}r \geqslant n\right\}, \qquad \text{where } \inf\varnothing = \tau. \tag{1.15}$$

Therefore, modifying $c^{(1.1)}$ if necessary, we deduce from (1.12):

$$\forall\, n \in \mathbb{N}^*, \qquad \frac{\left(C^{(1.1)}\right)^2}{2}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r \leqslant c^{(1.1)} + \frac{C^{(1.1)}}{2}\int_t^{\tau_n} \overline{Z}_r\,\mathrm{d}B_r. \tag{1.16}$$

Hence, modifying once again $c^{(1.1)}$ if necessary, we have for every $n \in \mathbb{N}^*$:

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\frac{\left(C^{(1.1)}\right)^2}{2}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r\right)\right] \\ &\qquad \leqslant c^{(1.1)}\,\mathbb{E}\left[\exp\left(\frac{C^{(1.1)}}{2}\int_t^{\tau_n} \overline{Z}_r\,\mathrm{d}B_r - \frac{\left(C^{(1.1)}\right)^2}{4}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r \right.\right. \\ &\qquad\qquad \left.\left. + \frac{\left(C^{(1.1)}\right)^2}{4}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r\right)\right]. \end{aligned} \tag{1.17}$$

From the Cauchy–Schwartz inequality, we deduce that for every $n \in \mathbb{N}^*$:

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\frac{\left(C^{(1.1)}\right)^2}{2}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r\right)\right] \\ &\quad \leqslant c^{(1.1)}\,\mathbb{E}\left[\exp\left(C^{(1.1)}\int_t^{\tau_n} \overline{Z}_r\,\mathrm{d}B_r - \frac{(C^{(1.1)})^2}{2}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r\right)\right]^{1/2} \\ &\qquad \times \mathbb{E}\left[\exp\left(\frac{\left(C^{(1.1)}\right)^2}{2}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r\right)\right]^{1/2}. \end{aligned} \tag{1.18}$$

Let us recall that:

$$\mathbb{E}\left[\exp\left(C^{(1.1)}\int_t^{\tau_n} \overline{Z}_r\,\mathrm{d}B_r - \frac{(C^{(1.1)})^2}{2}\int_t^{\tau_n} \left|\overline{Z}_r\right|^2 \mathrm{d}r\right)\right] = 1. \tag{1.19}$$

Using the Beppo-Levi theorem, we complete the proof of Lemma 1.1. □

Let us return to the proof of Theorem 1.1. Let us put:

$$\forall\, t \leqslant s \leqslant T, \qquad \overline{B}_s = B_s - C^{(1.1)} \int_t^{s\wedge\tau} \overline{Z}_r^* \,\mathrm{d}r. \tag{1.20}$$

From the Girsanov theorem, there exists a probability measure $\overline{\mathbb{P}}$ such that $(\overline{B}_s)_{t\leqslant s\leqslant T}$ is an $(\mathcal{F}_s)_{t\leqslant s\leqslant T}$-Brownian motion. The probability measure $\overline{\mathbb{P}}$ is given by:

$$\frac{\mathrm{d}\overline{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_s} = \exp\left(C^{(1.1)} \int_t^{s\wedge\tau} \overline{Z}_r \,\mathrm{d}B_r - \frac{\left(C^{(1.1)}\right)^2}{2} \int_t^{s\wedge\tau} \left|\overline{Z}_r\right|^2 \mathrm{d}r \right), \tag{1.21}$$

for all $t \leqslant s \leqslant T$. We denote by $\overline{\mathbb{E}}$ the expectation under $\overline{\mathbb{P}}$. Hence, noting from (1.12) that $(\int_t^{s\wedge\tau} \overline{Z}_r \,\mathrm{d}\overline{B}_r)_{t\leqslant s\leqslant T}$ is a bounded martingale, we deduce:

$$\overline{Y}_t = \overline{\mathbb{E}}\big[\mu\,\theta_i(\tau, X_\tau) + |\theta(\tau, X_\tau)|^2 + \big(1+\mu^2\big)C^{(1.1)}(\tau - t)\big]. \tag{1.22}$$

Hence, from (1.13), we deduce:

$$\begin{aligned} &\mu\,\theta_i(t,x) + |\theta(t,x)|^2 \\ &\qquad \leqslant \overline{\mathbb{E}}\big[\mu\,\theta^i(\tau, X_\tau) + |\theta(\tau, X_\tau)|^2 + \big(1+\mu^2\big)C^{(1.1)}(\tau - t)\big]. \end{aligned} \tag{1.23}$$

Choose now $\mu = 10\,QM_0$, where M_0 is given by Theorem 0.1.

Consider $(t_0, x_0) \in [0, T[\, \times \mathbb{R}^P$. Let us then adopt the following notations:

$$\forall\, 0 \leqslant r \leqslant R_0(t_0), \qquad \mathcal{Q}(r) = \mathcal{Q}_{(t_0,x_0)}(r). \tag{1.24}$$

Moreover, fix $R > 0$, such that $t_0 + 4R^2 \leqslant T$, and choose i as the integer of $\{1, \dots, Q\}$ such that:

$$\operatorname*{osc}_{\mathcal{Q}(2R)}(\theta_i) \geqslant \operatorname*{osc}_{\mathcal{Q}(2R)}(\theta_\ell), \qquad \ell \in \{1, \dots, Q\}. \tag{1.25}$$

We put:

$$\begin{cases} \text{for every } (t,x) \in \mathcal{Q}(2R), \\ w^+(t,x) = \big(\mu\,\theta_i + |\theta|^2\big)(t,x), \qquad w^-(t,x) = \big(-\mu\,\theta_i + |\theta|^2\big)(t,x), \\ \text{and} \qquad M^+ = \max_{\mathcal{Q}(2R)}(w^+), \qquad M^- = \max_{\mathcal{Q}(2R)}(w^-). \end{cases} \tag{1.26}$$

From the inequality (13.25) of Chapter 13 of Gilbarg and Trudinger [8], we know that for every $(s,y) \in \mathcal{Q}(2R)$,

$$M^+ - w^+(s,y) + M^- - w^-(s,y) \geqslant \frac{1}{2}\Big(\operatorname*{osc}_{\mathcal{Q}(2R)}(w^+) \vee \operatorname*{osc}_{\mathcal{Q}(2R)}(w^-) \Big). \tag{1.27}$$

Hence, from (1.27), we have (note that both inequalities may be true at the same time):

$$\mu_{P+1}(B^+) \geqslant \frac{1}{2}\,\mu_{P+1}\big(\mathcal{Q}(2R)\big) \quad \text{or} \quad \mu_{P+1}(B^-) \geqslant \frac{1}{2}\,\mu_{P+1}\big(\mathcal{Q}(2R)\big), \tag{1.28}$$

where:

$$\begin{cases} B^+ = \Big\{(s,y) \in \mathcal{Q}(2R),\ M^+ - w^+(s,y) \geqslant \dfrac{1}{4} \operatorname*{osc}_{\mathcal{Q}(2R)} (w^+)\Big\}, \\ B^- = \Big\{(s,y) \in \mathcal{Q}(2R),\ M^- - w^-(s,y) \geqslant \dfrac{1}{4} \operatorname*{osc}_{\mathcal{Q}(2R)} (w^-)\Big\}. \end{cases} \tag{1.29}$$

Let us assume that (1.28) holds with $+$.

Fix $(t,x) \in \mathcal{Q}(R)$, and put:

$$\begin{cases} \gamma = \inf\big\{s \geqslant t,\ (s,X_s) \in B^+\big\}, \\ \tau_{2R} = \inf\big\{s > t,\ (s,X_s) \in \partial\mathcal{Q}(2R)\big\}, \\ \tau = \gamma \wedge \tau_{2R}, \end{cases} \tag{1.30}$$

where X is given by (1.4).

Hence, from the inequality (1.23) applied to μ and to the stopping time τ defined in (1.30), there exists a constant $C^{(1.2)}$, only depending on Λ, λ, Q and T, such that:

$$\begin{aligned} w^+(t,x) \leqslant{}& M^+\,\overline{\mathbb{P}}\{\tau_{2R} < \gamma\} \\ &+ \Big(M^+ - \frac{1}{4} \operatorname*{osc}_{\mathcal{Q}(2R)} (w^+)\Big)\,\overline{\mathbb{P}}\{\gamma \leqslant \tau_{2R}\} + C^{(1.2)} R^2. \end{aligned} \tag{1.31}$$

Hence,

$$w^+(t,x) \leqslant M^+ - \frac{1}{4} \operatorname*{osc}_{\mathcal{Q}(2R)} (w^+)\,\overline{\mathbb{P}}\{\gamma \leqslant \tau_{2R}\} + C^{(1.2)} R^2. \tag{1.32}$$

We have to estimate $\overline{\mathbb{P}}\{\gamma \leqslant \tau_{2R}\}$. From Krylov and Safonov [15], we know that there exists $\eta^{(1.1)} > 0$, only depending on λ, Λ and P, such that:

$$\mathbb{P}\{\gamma \leqslant \tau_{2R}\} \geqslant \eta^{(1.1)}. \tag{1.33}$$

Noting from (1.20) and (1.21) that, for all $t \leqslant s \leqslant T$,

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\overline{\mathbb{P}}}\bigg|_{\mathcal{F}_s} = \exp\bigg(-C^{(1.1)} \int_t^{s\wedge\tau} \overline{Z}_r\,\mathrm{d}\overline{B}_r - \frac{\big(C^{(1.1)}\big)^2}{2} \int_t^{s\wedge\tau} \big|\overline{Z}_r\big|^2\,\mathrm{d}r\bigg), \tag{1.34}$$

we have:

$$\begin{aligned} \mathbb{P}\{\gamma \leqslant \tau_{2R}\} = \overline{\mathbb{E}}\bigg[\exp\bigg(&-C^{(1.1)} \int_t^{\tau} \overline{Z}_r\,\mathrm{d}\overline{B}_r \\ &- \frac{1}{2}\,\big(C^{(1.1)}\big)^2 \int_t^{\tau} \big|\overline{Z}_r\big|^2\,\mathrm{d}r\bigg)\,\mathbf{1}_{\{\gamma\leqslant\tau_{2R}\}}\bigg]. \end{aligned} \tag{1.35}$$

Now, applying Proposition 2.1 of Kobylanski [12] to estimate the process $\overline{Y}$ in (1.12), we deduce that there exists a constant $C^{(1.3)}$, only depending on Λ, λ, Q and T, such that:

$$\left|\int_t^\tau \overline{Z}_r \,\mathrm{d}\overline{B}_r\right| \leqslant C^{(1.3)}. \tag{1.36}$$

This proves that there exists $\eta^{(1.2)} > 0$, only depending on Λ, λ, P, Q and T, such that:

$$\overline{\mathbb{P}}\{\gamma \leqslant \tau_{2R}\} \geqslant \eta^{(1.2)}. \tag{1.37}$$

Hence, from (1.32), we can find $0 < \eta^{(1.3)} < 1$, only depending on Λ, λ, P, Q and T, such that:

$$\operatorname*{osc}_{\mathcal{Q}(R)}(w^+) \leqslant \left(1 - \eta^{(1.3)}\right) \operatorname*{osc}_{\mathcal{Q}(2R)}(w^+) + C^{(1.2)} R^2. \tag{1.38}$$

In the same way, applying (1.23) to $-\mu$ and to τ, we prove that (1.38) holds with w^- instead of w^+ as soon as (1.28) holds with $-$.

From Lemma 13.5 of Chapter 13 of Gilbarg and Trudinger [8], we conclude that there exist two constants $C^{(1.4)}$ and $\alpha > 0$, only depending on Λ, λ, P, Q and T, such that for every $R \leqslant R_0(t_0)$:

$$\forall\, \ell \in \{1, \dots, Q\}, \qquad \operatorname*{osc}_{\mathcal{Q}(R)}(\theta_\ell) \leqslant C^{(1.4)} \left(\left(\frac{R}{R_0(t_0)}\right)^\alpha w_0 + R R_0(t_0) \right), \tag{1.39}$$

where:

$$w_0 = \max\Big\{ \operatorname*{osc}_{\mathcal{Q}(R_0(t_0))} \left(\varepsilon\, \mu\, \theta_\ell + |\theta|^2\right),\ \ell = 1, \dots, Q,\ \varepsilon = -1, 1 \Big\}. \tag{1.40}$$

This completes the proof of Theorem 1.1. □

Let us now deduce from the former theorem the following interior estimate of Hölder type of the function θ:

Theorem 1.2. *There exists a constant $\Gamma^{(1.2)}$, only depending on Λ, λ, P, Q and T, such that for every $t \in [0, T[$, the following inequality holds for all (r, x), $(s, y) \in [0, t] \times \mathbb{R}^P$:*

$$|\theta(s, y) - \theta(r, x)| \leqslant \frac{\Gamma^{(1.2)}}{(T - t)^{\alpha/2}} \left(|y - x|^\alpha + |s - r|^{\alpha/2}\right), \tag{1.41}$$

where α is given by Theorem 1.1.

Proof. Consider $t \in [0, T[$ as well as (r, x), $(s, y) \in [0, t] \times \mathbb{R}^P$, $r \leqslant s$.

Let us assume for the moment that $\max_{i=1,\dots,P} |y_i - x_i|^2 \leqslant T - t$.

Letting $R_1 = \max_{i=1,\dots,P} |y_i - x_i|$, and noting that $R_1^2 \leqslant T - r$, we deduce from Theorem 0.1 and Theorem 1.1 applied to the cylinder $\mathcal{Q}_{(r,x)}(R_1)$:

$$|\theta_i(r, y) - \theta_i(r, x)| \leqslant \Gamma^{(1.1)} \left(C^{(1.5)} \frac{|y - x|^\alpha}{(T - t)^{\alpha/2}} + \sqrt{T - r}\, |y - x| \right), \tag{1.42}$$

for every $i \in \{1, \dots, Q\}$, where $C^{(1.5)}$ only depends on Λ, Q and T.

Now, letting $R_2^2 = s - r$, and noting that $R_2^2 \leqslant T - r$, we deduce from Theorem 1.1 applied to the cylinder $\mathcal{Q}_{(r,y)}(R_2)$:

$$|\theta_i(s, y) - \theta_i(r, y)| \leqslant \Gamma^{(1.1)}\left(C^{(1.5)}\left(\frac{s-r}{T-t}\right)^{\alpha/2} + \sqrt{T-r}\,\sqrt{s-r}\right), \quad (1.43)$$

for every $i \in \{1, \dots, Q\}$. Summing (1.42) and (1.43), we deduce that there exists a constant $\Gamma^{(1.2)}$, only depending on Λ, λ, P, Q and T, such that (1.41) holds as soon as $\max_{i=1,\dots,P} |y_i - x_i|^2 \leqslant T - t$.

Modifying $\Gamma^{(1.2)}$ if necessary, we prove thanks to Theorem 0.1 that it still holds for all $(r, x), (s, y) \in [0, t] \times \mathbb{R}^P$. □

Making use of the assumption (A.3), we deduce the following global estimate of Hölder type of the solution θ:

Theorem 1.3. *There exists a constant $\Gamma^{(1.3)}$, only depending on L, Λ, λ, P, Q and T, such that:*

$$\forall\,(t, x),\ (s, y) \in [0, T] \times \mathbb{R}^P,$$
$$|\theta(s, y) - \theta(t, x)| \leqslant \Gamma^{(1.3)}\big(|x - y|^{\alpha'} + |t - s|^{\alpha'/2}\big), \quad (1.44)$$

where $\alpha' = \alpha \wedge \alpha_0$.

Proof. Consider (t, x), $(s, y) \in [0, T] \times \mathbb{R}^P$, such that $t \leqslant s$.

Following the notations of Theorem 1.1, we assume for the moment that there exists a constant $C^{(1.6)}$, only depending on L, Λ, Q and T, such that:

$$w_0(t, x) \leqslant C^{(1.6)}\big(R_0(t)\big)^{\alpha_0}. \quad (1.45)$$

Then, letting $R_0 = R_0(t)$ and $\mathcal{Q}(R) = \mathcal{Q}_{(t,x)}(R)$, we have from Theorem 0.1 and Theorem 1.1:

$$\forall\, 0 \leqslant R \leqslant R_0,\ \forall\, i \in \{1, \dots, Q\},$$
$$\operatorname*{osc}_{\mathcal{Q}(R)}(\theta_i) \leqslant \Gamma^{(1.1)}\big(C^{(1.6)} R^{\alpha'} R_0^{\alpha_0 - \alpha'} + R R_0\big). \quad (1.46)$$

In particular, following the proof of Theorem 1.2, we deduce that there exists a constant $\Gamma^{(1.3)}$, only depending on L, Λ, λ, P, Q and T, such that (1.44) holds as soon as $\max_{i=1,\dots,P} |x_i - y_i|^2 \leqslant T - t$.

Let us assume that $\max_{i=1,\dots,P} |x_i - y_i|^2 > T - t$. Then, applying (1.44) a first time to the couples (t, x) and (T, x), and a second one to the couples (s, y) and (T, y), we deduce:

$$\begin{aligned}
&|\theta(t, x) - \theta(s, y)| \\
&\quad \leqslant |\theta(t, x) - H(x)| + |H(x) - H(y)| + |\theta(s, y) - H(y)| \\
&\quad \leqslant 2\,\Gamma^{(1.3)}(T - t)^{\alpha'/2} + L|x - y|^{\alpha_0} \\
&\quad \leqslant 2\,\Gamma^{(1.3)}|x - y|^{\alpha'} + L|x - y|^{\alpha_0}.
\end{aligned} \quad (1.47)$$

Modifying $\Gamma^{(1.3)}$ if necessary, we deduce that (1.44) holds as soon as $|x-y| \leqslant 1$. Thanks to Theorem 0.1, we show that it actually holds for all $(s,y),(t,x) \in [0,T] \times \mathbb{R}^P$. $\square$

Hence, we just have to prove the following result:

Lemma 1.2. *There exists a constant $C^{(1.6)}$, only depending on L, Λ, Q and T, such that for every $(t,x) \in [0,T[\times \mathbb{R}^P$:*

$$w_0(t,x) \leqslant C^{(1.6)} \big(R_0(t)\big)^{\alpha_0}. \tag{1.48}$$

Proof. Consider $(t,x) \in [0,T] \times \mathbb{R}^P$. We recall that $(X_s)_{t \leqslant s \leqslant T}$ is given by:

$$X_s = x + \int_t^s \sigma\big(r, X_r, \theta(r, X_r)\big)\,\mathrm{d}B_r. \tag{1.49}$$

We also recall that, for every $n \in \mathbb{N}$, the process $(X_s, Y_s, Z_s)_{t \leqslant s \leqslant t^n}$ satisfies the following FBSDE:

$$\begin{cases} X_s = x + \displaystyle\int_t^s b(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}B_r^n, \\ Y_s = \theta(t^n, X_{t^n}) + \displaystyle\int_s^{t^n} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t^n} Z_r\,\mathrm{d}B_r^n, \\ \text{for all } s \in [t, t^n], \text{ and } \mathbb{E}^n\left[\displaystyle\int_t^{t^n} \big(|X_s|^2 + |Y_s|^2 + |Z_s|^2\big)\,\mathrm{d}s\right] < \infty. \end{cases} \tag{1.50}$$

where for every $n \in \mathbb{N}$, t^n, B^n and $\mathbb{E}^n$ have been defined in (0.11), (0.12) and (0.13).

Then, thanks to Theorem 0.1 and to the inequality (0.15), we prove that there exists a constant $c^{(1.3)}$, only depending on Λ and T such that:

$$\forall\, n \in \mathbb{N}, \qquad \mathbb{E}^n\left[\int_t^{t^n} |Z_s|^2\,\mathrm{d}s\right] \leqslant c^{(1.3)}. \tag{1.51}$$

Moreover, from Itô's formula applied to the process $(|Y_s - H(x)|^2)_{t \leqslant s \leqslant t^n}$, there exists a constant $C^{(1.7)}$, only depending on Λ and T such that for every $n \in \mathbb{N}$:

$$\begin{aligned} |\theta(t,x) - H(x)|^2 &\leqslant C^{(1.7)}\big(\mathbb{E}^n\big[|\theta(t^n, X_{t^n}) - H(x)|^2\big] + (T-t)\big) \\ &\leqslant C^{(1.7)}\big(\mathbb{E}^n\big[|H(X_T) - H(x)|^2\big] \\ &\qquad + \mathbb{E}^n\big[\mathbf{1}_{\{t^n < T\}}|\theta(t^n, X_{t^n}) - H(x)|^2\big] + (T-t)\big). \end{aligned} \tag{1.52}$$

Hence, from Assumption (A) and thanks to Theorem 0.1, there exists a constant $C^{(1.8)}$ (whose value may change from one inequality to another), only depending on L, Λ and T, such that for every $n \in \mathbb{N}^*$:

$$\begin{aligned} &|\theta(t,x) - H(x)|^2 \\ &\quad \leqslant C^{(1.8)}\big(\mathbb{E}^n\big[|X_T - x|^{2\alpha_0}\big] + \mathbb{P}^n\{t^n < T\} + (T-t)\big) \\ &\quad \leqslant C^{(1.8)}\left(\mathbb{E}^n\big[|X_T - x|^2\big]^{\alpha_0} + \frac{1}{n^2}\,\mathbb{E}^n\left[\sup_{t \leqslant s \leqslant T} |X_s - x|^2\right] + (T-t)\right). \end{aligned} \tag{1.53}$$

Noting that $(X_s)_{t\leqslant s\leqslant T}$ satisfies, for all $s\in[t,T]$,

$$X_s = x + \int_t^{s\wedge t^n} b(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_t^s \sigma(r,X_r,Y_r)\,\mathrm{d}B_r^n, \tag{1.54}$$

we deduce from Theorem 0.1 and (1.53) that for every $n\in\mathbb{N}^*$:

$$|\theta(t,x)-H(x)|^2 \leqslant C^{(1.8)}\Big(1+\frac{1}{n^2}\Big)(T-t)^{\alpha_0}. \tag{1.55}$$

Hence, letting $n\to+\infty$ and applying the property (A.3), we have for every $(s,y)\in\mathcal{Q}_{(t,x)}(R_0(t))$:

$$|\theta(t,x)-\theta(s,y)| \leqslant C^{(1.8)}(T-t)^{\alpha_0/2} = C^{(1.8)}\big(R_0(t)\big)^{\alpha_0}. \tag{1.56}$$

From the definition of w_0, this completes the proof of Lemma 1.2. □

2 Estimates of the Gradient of the Solution

This section is devoted to the proof of the gradient estimate required in the article of Delarue [6] and proved in Chapter VII of Ladyzhenskaya et al. [18]. To this aim, we now assume that the function θ belongs to the space $C^{1,2}([0,T]\times\mathbb{R}^P,\mathbb{R}^Q)\cap L^\infty([0,T]\times\mathbb{R}^P,\mathbb{R}^Q)$ and that the coefficients satisfy the followings:

Assumption (A'). We say that the functions b, f, H and σ satisfy Assumption (A') if there exist five constants $\gamma>0$, k, L, $\lambda>0$ and Λ, such that they satisfy Assumption (A) with respect to the constants $\alpha_0=1$, L, λ and Λ, as well as the following properties:

(A.5) $\forall\,(t,x,y)\in[0,T]\times\mathbb{R}^P\times\mathbb{R}^Q$, $\forall\,(x',y')\in\mathbb{R}^P\times\mathbb{R}^Q$,

$$|\sigma(t,x',y')-\sigma(t,x,y)| \leqslant k\,\big(|x'-x|+|y'-y|\big).$$

(A.6) The function σ is differentiable with respect to x and y and its derivatives with respect to x and y are γ-Hölder in x and y, uniformly in t.

In particular, from the choice of α_0, the function H is L-Lipschitzian. Moreover, the function θ satisfies Theorems 0.1, 1.2 and 1.3 with $\alpha'=\alpha$, α being given by Theorem 1.1.

Representation of the solution θ. Consider a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, a filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ satisfying the usual conditions, and an $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$-Brownian motion, denoted by $(B_t)_{0\leqslant t\leqslant T}$.

Moreover, for a fixed real $m>0$, consider on the one hand a bounded open set $\mathcal{O}\subset[0,T]\times\mathbb{R}^P$ such that:

$$[0,T] \times \overline{B}_P\Big(0, m+\sqrt{T}\Big) \subset \mathcal{O}, \tag{2.0.1}$$

and on the other one a function $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^P, \mathbb{R}^Q)$, with compact support, such that:

$$\forall\, (t,x) \in \mathcal{O}, \qquad \varphi(t,x) = \theta(t,x).$$

Hence, thanks to Assumption (A'), we can consider, for every $(t,x) \in [0,T] \times \mathbb{R}^P$, the unique solution, still denoted by $(X_s^{t,x})_{t \leqslant s \leqslant T}$, of the following equation:

$$\forall\, s \in [t,T], \qquad X_s^{t,x} = x + \int_t^s \sigma\big(r, X_r^{t,x}, \varphi(r, X_r^{t,x})\big)\,\mathrm{d}B_r. \tag{2.0.2}$$

Thus, following Proposition 0.1, for every $(t,x) \in [0,T] \times \mathbb{R}^P$, for every $(\mathcal{F}_s)_{t \leqslant s \leqslant T}$ stopping time τ satisfying:

$$\mathbb{P}\Big\{\sup_{t \leqslant s \leqslant \tau} |X_s^{t,x}| \leqslant m + \sqrt{T}\Big\} = 1, \tag{2.0.3}$$

the process $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{t \leqslant s \leqslant \tau}$, given by, for all $t \leqslant s \leqslant T$,

$$Y_s^{t,x} = \varphi(s, X_s^{t,x}), \qquad Z_s^{t,x} = \nabla_x \varphi(s, X_s^{t,x})\, \sigma(s, X_s^{t,x}, Y_s^{t,x}), \tag{2.0.4}$$

satisfies the following FBSDE:

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x})\,\mathrm{d}B_r, \\ Y_s^{t,x} = \varphi(\tau, X_\tau^{t,x}) + \displaystyle\int_s^\tau e(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,\mathrm{d}r - \int_s^\tau Z_r^{t,x}\,\mathrm{d}B_r, \\ \text{for all } s \in [t,\tau], \text{ and } \mathbb{E}\Big[\displaystyle\int_t^\tau \big(|X_s^{t,x}|^2 + |Y_s^{t,x}|^2 + |Z_s^{t,x}|^2\big)\,\mathrm{d}s\Big] < \infty. \end{cases} \tag{2.0.5}$$

Remark 2.1. Note that, for all $t \leqslant s \leqslant \tau$,

$$Y_s^{t,x} = \theta\big(s, X_s^{t,x}\big), \qquad Z_s^{t,x} = \nabla_x \theta\big(s, X_s^{t,x}\big)\, \sigma\big(s, X_s^{t,x}, Y_s^{t,x}\big). \tag{2.0.6}$$

This explains why we have kept the notations of Sect. 1.

Let us also recall from Theorem 4.6.5 of Chapter IV of Kunita [17] that, under Assumption (A'), for almost every $\omega \in \Omega$, the map $(t,x) \mapsto X^{t,x} \in C([0,T] \times \mathbb{R}^P, \mathbb{R}^P)$ is differentiable with respect to x. Moreover, denoting by $\partial_i X^{t,x}$ the partial derivative with respect to x_i, the following equation holds for every $i \in \{1, \dots, P\}$:

$$\partial_i X_s^{t,x} = e_i + \int_t^s \Big(\sigma'_x(r, X_r^{t,x}, Y_r^{t,x})\, \partial_i X_r^{t,x} \\ + \big(\sigma'_y \nabla_x \varphi\big)(r, X_r^{t,x}, Y_r^{t,x})\, \partial_i X_r^{t,x}\Big)\,\mathrm{d}B_r, \tag{2.0.7}$$

for all $s \in [t,T]$, and the map $(t,x) \mapsto \nabla_x X^{t,x} = (\partial_1 X^{t,x}, \dots, \partial_P X^{t,x}) \in C([0,T] \times \mathbb{R}^P, \mathbb{R}^{P\times P})$ is, thanks to Kolmogorov's Lemma, almost surely continuous and satisfies for every compact set $\kappa \subset [0,T] \times \mathbb{R}^P$:

$$\forall p \geqslant 1, \qquad \mathbb{E}\left[\sup_{(t,x)\in\kappa}\left(\sup_{t\leqslant s\leqslant T} |\nabla_x X_s^{t,x}|^p\right)\right] < \infty. \tag{2.0.8}$$

Finally, we adopt the following notations:

Notations. For every $(t,x) \in [0,T] \times \mathbb{R}^P$ and for every $0 < R \leqslant \sqrt{T-t}$, let:

$$\mathcal{C}_{(t,x)}(R) = \left[t, t+R^2\right[\times B_P(x,R). \tag{2.0.9}$$

Fix now for the whole section $(t_0,x_0) \in [0,T[\times B_P(0,m)$ and $0 < R \leqslant (T-t_0)^{1/2}$. Set:

$$\begin{cases} \mathcal{C} = \mathcal{C}_{(t_0,x_0)}(R), \\ \forall\,(t,x) \in \mathcal{C},\ \tau^{t,x} = \tau_{\mathcal{C}}^{t,x} = \inf\{s \geqslant t,\ (s, X_s^{t,x}) \notin \mathcal{C}\}. \end{cases} \tag{2.0.10}$$

Note from the choice of x_0 that:

$$\forall\, 0 \leqslant R \leqslant \sqrt{T-t_0}, \qquad \overline{\mathcal{C}} \subset [0,T] \times B_P\left(0, m+\sqrt{T}\right). \tag{2.0.11}$$

Moreover, consider also $t_0 \leqslant u_0 \leqslant u \leqslant t_0 + R^2$, $z \in \mathbb{R}^P$ and $\varrho > 0$ such that $\varrho + |z - x_0| \leqslant R$. Set for every $n \in \mathbb{N}^*$:

$$\begin{cases} \mathcal{D} = [u_0, u[\times B_P(z,\varrho), \\ \mathcal{D}_n = [u_0, u_0 + (1-1/n)(u-u_0)[\times B_P\big(z, (1-1/n)\varrho\big). \end{cases} \tag{2.0.12}$$

Note that for every $n \in \mathbb{N}^*$, $\mathcal{D}_n \subset \mathcal{D}_{n+1} \subset \mathcal{D} \subset \mathcal{C}$.

Finally, let for every $(t,x) \in [0,T] \times \mathbb{R}^P$ and for every $n \in \mathbb{N}^*$:

$$\begin{cases} \zeta^{t,x} = \inf\{s \geqslant t,\ (s, X_s^{t,x}) \notin \mathcal{D}\}, \\ \zeta^{n,t,x} = \inf\{s \geqslant t,\ (s, X_s^{t,x}) \notin \mathcal{D}_n\}. \end{cases} \tag{2.0.13}$$

Actually, this section is divided into three parts. In the first one, we establish basic estimates of the processes Z and $\nabla_x X$. The second one is devoted to the proof of the integration by parts formula required to estimate the gradient of θ. This estimate is given in the third and last part of this section.

2.1 Estimates of Z and $\nabla_x X$

This first subsection is devoted to the proof of basic estimates of the processes Z and $\nabla_x X$.

Lemma 2.1. *There exists a constant $\Gamma^{(2.1)}$, only depending on Λ and T, such that for every $0 < R \leqslant \sqrt{T-t}$ and for every $(t,x) \in \mathcal{C}$:*

$$\forall\, s > t, \qquad \mathbb{E}\left[\frac{\left|X^{t,x}_{s\wedge\tau^{t,x}} - x\right|^2}{(s\wedge\tau^{t,x} - t)^{1/2}}\right] \leqslant \Gamma^{(2.1)}. \tag{2.1.1}$$

Proof. Consider $0 < R \leqslant \sqrt{T-t}$ and $(t,x) \in \mathcal{C}$. Note that we omit to specify the dependence upon (t,x) of $X^{t,x}$ and $\tau^{t,x}$.

Let $\varepsilon > 0$. Considering the semimartingale

$$\left(\frac{|X_s - x|^2}{\varepsilon + (s-t)^{1/2}}\right)_{t\leqslant s\leqslant T},$$

we have from Itô's formula:

$$\forall\, s > t, \qquad \mathbb{E}\left[\frac{|X_{s\wedge\tau} - x|^2}{\varepsilon + (s\wedge\tau - t)^{1/2}}\right] \leqslant \mathbb{E}\left[\int_t^{s\wedge\tau} \frac{|\sigma(r, X_r, Y_r)|^2}{\varepsilon + (r-t)^{1/2}}\,\mathrm{d}r\right]. \tag{2.1.2}$$

Hence, applying Theorem 0.1 and letting $\varepsilon \to 0$, we complete the proof. □

Proposition 2.1. *There exists three constants $0 < \beta < 1$, $c^{(2.1)}$ and $\Gamma^{(2.2)}$, only depending on L, Λ, λ, P, Q and T, such that for every $R \leqslant c^{(2.1)}$ and for every $(t,x) \in \mathcal{C}$:*

$$\mathbb{E}\left[\int_t^{\tau^{t,x}} \frac{|Z^{t,x}_s|^2}{(s-t)^\beta}\,\mathrm{d}s\right] \leqslant \Gamma^{(2.2)}. \tag{2.1.3}$$

Proof. Consider $0 < R \leqslant \sqrt{T-t}$ and $(t,x) \in \mathcal{C}$. Once again, we omit to specify the dependence of $X^{t,x}$, $Y^{t,x}$, $Z^{t,x}$ and $\tau^{t,x}$ upon (t,x).

Let $0 < \beta < 1$ and $\varepsilon > 0$. Considering the semimartingale

$$\left(\frac{|Y_s - Y_t|^2}{\varepsilon + (s-t)^\beta}\right)_{t\leqslant s\leqslant \tau},$$

we have from Itô's formula:

$$\begin{aligned}
\mathbb{E}\left[\int_t^\tau \frac{|Z_s|^2}{\varepsilon + (s-t)^\beta}\,\mathrm{d}s\right] \leqslant \mathbb{E}&\left[\frac{|Y_\tau - Y_t|^2}{\varepsilon + (\tau - t)^\beta}\right] \\
&+ 2\,\mathbb{E}\left[\int_t^\tau \frac{\langle Y_s - Y_t, e(s, X_s, Y_s, Z_s)\rangle}{\varepsilon + (s-t)^\beta}\,\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_t^\tau \frac{|Y_s - Y_t|^2}{(s-t)^{1-\beta}(\varepsilon + (s-t)^\beta)^2}\,\mathrm{d}s\right].
\end{aligned} \tag{2.1.4}$$

From Theorem 1.3 and the definition of τ, we know that there exists a constant $C^{(2.1)}$, only depending on L, λ, Λ, P, Q and T, such that:

$$\forall\, t \leqslant s \leqslant \tau, \qquad |Y_s - Y_t| \leqslant C^{(2.1)} R^\alpha. \tag{2.1.5}$$

Hence, again from Theorems 0.1 and 1.3, there exists a constant $C^{(2.2)}$, only depending on L, λ, Λ, P, Q and T, such that:

$$\begin{aligned}\mathbb{E}\left[\int_t^\tau \frac{|Z_s|^2}{\varepsilon+(s-t)^\beta}\,\mathrm{d}s\right] \leqslant C^{(2.2)}\Bigg(&\mathbb{E}\left[\frac{(\tau-t)^\alpha+|X_\tau-x|^{2\alpha}}{\varepsilon+(\tau-t)^\beta}\right]\\ &+R^\alpha\,\mathbb{E}\left[\int_t^\tau \frac{1+|Z_s|^2}{\varepsilon+(s-t)^\beta}\,\mathrm{d}s\right]\\ &+\mathbb{E}\left[\int_t^\tau \frac{(s-t)^\alpha+|X_s-x|^{2\alpha}}{(s-t)^{1-\beta}(\varepsilon+(s-t)^\beta)^2}\,\mathrm{d}s\right]\Bigg).\end{aligned} \tag{2.1.6}$$

Choose:

$$\beta=\frac{\alpha}{4}. \tag{2.1.7}$$

Hence, modifying $C^{(2.2)}$ if necessary, we deduce:

$$\begin{aligned}\mathbb{E}\left[\int_t^\tau \frac{|Z_s|^2}{\varepsilon+(s-t)^\beta}\,\mathrm{d}s\right] \leqslant C^{(2.2)}\Bigg(&1+\mathbb{E}\left[\frac{|X_\tau-x|^2}{\varepsilon+(\tau-t)^{1/4}}\right]^\alpha\\ &+R^\alpha\,\mathbb{E}\left[\int_t^\tau \frac{|Z_s|^2}{\varepsilon+(s-t)^\beta}\,\mathrm{d}s\right]\\ &+\int_t^{t_0+R^2} \frac{1}{(s-t)^{1-\alpha/4}}\,\mathbb{E}\left[\frac{|X_{s\wedge\tau}-x|^2}{(s\wedge\tau-t)^{1/2}}\right]^\alpha \mathrm{d}s\Bigg).\end{aligned} \tag{2.1.8}$$

Choosing a small enough R, and letting $\varepsilon \to 0$, we complete the proof. □

Proposition 2.2. *For every $p \geqslant 1$, there exist two constants $c_p^{(2.2)}$ and $\Gamma_p^{(2.3)}$, only depending on k, L, λ, Λ, p, P, Q and T, such that for every $R \leqslant c_p^{(2.2)}$ and for every $(t,x) \in \mathcal{C}$:*

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant \tau^{t,x}} |\nabla_x X_s^{t,x}|^{2p}\right] \leqslant \Gamma_p^{(2.3)}. \tag{2.1.9}$$

Proof. We keep the notations of the statement, but we omit to specify the dependence upon (t,x) of $X^{t,x}$, $Y^{t,x}$, $Z^{t,x}$ and $\tau^{t,x}$.

From (2.0.7), we know that for every $i \in \{1,\dots,P\}$:

$$\begin{aligned}\forall\, s\in[t,T],\qquad \partial_i X_s = e_i + \int_t^s \Big(&\sigma'_x(r,X_r,Y_r)\,\partial_i X_r\\ &+\big(\sigma'_y Z_r \sigma^{-1}\big)(r,X_r,Y_r)\,\partial_i X_r\Big)\,\mathrm{d}B_r.\end{aligned} \tag{2.1.10}$$

Consider $\mu > 0$ and $i \in \{1,\dots,P\}$. Thanks to the system (2.0.5), we deduce from Itô's formula that for every $t \leqslant s \leqslant \tau$:

$$
\begin{aligned}
&\mathrm{d}\big(\exp(-\mu\,|Y_s - Y_t|^2)\,|\partial_i X_s|^{2p}\big) = \exp\big(-\mu\,|Y_s - Y_t|^2\big)\\
&\times \bigg\{|\partial_i X_s|^{2p}\Big(\big(2\,\mu\,\langle Y_s - Y_t, e\rangle + 2\,\mu^2\big|Z_s^T(Y_s - Y_t)\big|^2 - \mu\,|Z_s|^2\big)\,\mathrm{d}s\\
&\quad - 2\,\mu\,\big\langle Y_s - Y_t, Z_s\,\mathrm{d}B_s\big\rangle\Big)\\
&\quad + p\,|\partial_i X_s|^{2(p-2)}\Big(2\,|\partial_i X_s|^2\big\langle \partial_i X_s, \big((\sigma'_x + \sigma'_y Z_s\sigma^{-1})\,\partial_i X_s\big)\,\mathrm{d}B_s\big\rangle \qquad (2.1.11)\\
&\quad + |\partial_i X_s|^2\big|\big(\sigma'_x + \sigma'_y Z_s\sigma^{-1}\big)\,\partial_i X_s\big|^2\,\mathrm{d}s\\
&\quad + 2\,(p-1)\big|\big((\sigma'_x + \sigma'_y Z_s\sigma^{-1})\,\partial_i X_s\big)^T\,\partial_i X_s\big|^2\,\mathrm{d}s\\
&\quad - 4\,\mu\,|\partial_i X_s|^2\big\langle Z_s^T(Y_s - Y_t), \big((\sigma'_x + \sigma'_y Z_s\sigma^{-1})\partial_i X_s\big)^T\,\partial_i X_s\big\rangle\,\mathrm{d}s\Big)\bigg\}.
\end{aligned}
$$

where, to simplify the notations, we have written e, σ'_x, σ'_y and σ^{-1} instead of $e(s, X_s, Y_s, Z_s)$, $\sigma'_x(s, X_s, Y_s)$, $\sigma'_y(s, X_s, Y_s)$ and $\sigma^{-1}(s, X_s, Y_s)$.

Hence, thanks to (2.0.8), we deduce that for every $s \in [t, T]$:

$$
\begin{aligned}
&\mathbb{E}\big[\exp\big(-\mu|Y_{s\wedge\tau} - Y_t|^2\big)\,|\partial_i X_{s\wedge\tau}|^{2p}\big]\\
&\quad + \mu\,\mathbb{E}\bigg[\int_t^{s\wedge\tau} \exp\big(-\mu\,|Y_r - Y_t|^2\big)\,|\partial_i X_r|^{2p}\,|Z_r|^2\,\mathrm{d}r\bigg]\\
&= 1 + \mathbb{E}\bigg[\int_t^{s\wedge\tau} \mathrm{d}r\bigg(\exp\big(-\mu\,|Y_r - Y_t|^2\big)\\
&\quad \times \Big\{|\partial_i X_r|^{2p}\Big(2\,\mu\,\langle Y_r - Y_t, e\rangle + 2\,\mu^2\big|Z_r^T(Y_r - Y_t)\big|^2\Big) \qquad (2.1.12)\\
&\quad + p\,|\partial_i X_r|^{2(p-2)}\Big(|\partial_i X_r|^2\big|\big(\sigma'_x + \sigma'_y Z_r\sigma^{-1}\big)\,\partial_i X_r\big|^2\\
&\quad + 2(p-1)\,\big|\big((\sigma'_x + \sigma'_y Z_r\sigma^{-1})\,\partial_i X_r\big)^T\partial_i X_r\big|^2\\
&\quad - 4\,\mu\,|\partial_i X_r|^2\Big\langle Z_r^T(Y_r - Y_t), \big((\sigma'_x + \sigma'_y Z_r\sigma^{-1})\,\partial_i X_r\big)^T\partial_i X_r\Big\rangle\Big)\Big\}\bigg)\bigg].
\end{aligned}
$$

Hence, from Theorem 0.1 and (2.1.5), there exists a constant $C_p^{(2.3)}$, only depending on k, L, λ, Λ, p, P, Q and T, such that for every $s \in [t, T]$:

$$
\begin{aligned}
&\mathbb{E}\big[\exp\big(-\mu\,|Y_{s\wedge\tau} - Y_t|^2\big)\,|\partial_i X_{s\wedge\tau}|^{2p}\big]\\
&\qquad + \mu\,\mathbb{E}\bigg[\int_t^{s\wedge\tau} \exp\big(-\mu|Y_r - Y_t|^2\big)\,|\partial_i X_r|^{2p}\,|Z_r|^2\,\mathrm{d}r\bigg]\\
&\quad \leqslant 1 + C_p^{(2.3)}\,\mathbb{E}\bigg[\int_t^{s\wedge\tau} \exp(-\mu\,|Y_r - Y_t|^2)\,|\partial_i X_r|^{2p} \qquad (2.1.13)\\
&\qquad \times \Big(\big(1 + \mu^2 R^{2\alpha}\big) + \big(1 + \mu R^{\alpha} + \mu^2 R^{2\alpha}\big)|Z_r|^2\Big)\,\mathrm{d}r\bigg].
\end{aligned}
$$

Choose $\mu = 3\,C_p^{(2.3)}$ as well as R such that:

$$\mu R^{\alpha} + \mu^2 R^{2\alpha} \leqslant 1. \tag{2.1.14}$$

Then, for every $s \in [t,T]$:

$$\begin{aligned} &\mathbb{E}\Big[\exp\big(-\mu|Y_{s\wedge\tau} - Y_t|^2\big)\,|\partial_i X_{s\wedge\tau}|^{2p}\Big] \\ &\qquad + C_p^{(2.3)}\,\mathbb{E}\Big[\int_t^{s\wedge\tau} \exp\big(-\mu\,|Y_r - Y_t|^2\big)\,|\partial_i X_r|^{2p}\,|Z_r|^2\,\mathrm{d}r\Big] \\ &\quad \leqslant 1 + 2\,C_p^{(2.3)}\,\mathbb{E}\Big[\int_t^{s\wedge\tau} \exp\big(-\mu|Y_r - Y_t|^2\big)\,|\partial_i X_r|^{2p}\,\mathrm{d}r\Big]. \end{aligned} \tag{2.1.15}$$

Hence, from Gronwall's lemma, there exist two constants $C_p^{(2.4)}$ (whose value may change from one inequality to another) and $\eta_p^{(2.1)}$, only depending on k, L, λ, Λ, p, P, Q and T, such that for every $R \leqslant \eta_p^{(2.1)}$:

$$\begin{aligned} &\mathbb{E}\Big[\exp\big(-\mu\,|Y_{s\wedge\tau} - Y_t|^2\big)|\partial_i X_{s\wedge\tau}|^{2p}\Big] \\ &\qquad + \mathbb{E}\Big[\int_t^{s\wedge\tau}\Big(\exp\big(-\mu\,|Y_r - Y_t|^2\big)\,|\partial_i X_r|^{2p}|Z_r|^2\Big)\,\mathrm{d}r\Big] \leqslant C_p^{(2.4)}, \end{aligned} \tag{2.1.16}$$

for all $s \in [t,T]$. Thanks to Theorem 0.1, we deduce that (2.1.16) holds with $\mu = 0$.

Hence, applying Doob's inequalities to the martingale $(\partial_i X_{s\wedge\tau})_{t\leqslant s\leqslant T}$, we deduce that for every $R \leqslant \eta_p^{(2.1)}$:

$$\mathbb{E}\Big[\sup_{t\leqslant s\leqslant\tau} |\partial_i X_s|^{2p}\Big] \leqslant C_p^{(2.4)}. \quad \square \tag{2.1.17}$$

Under the notations (2.0.12) and (2.0.13), we claim that:

Lemma 2.2. *For every $m > 0$, there exist two constants $c_m^{(2.3)}$ and $\Gamma_m^{(2.4)}$, only depending on k, L, λ, Λ, m, P, Q and T, such that for every $R \leqslant c_m^{(2.3)}$:*

$$\forall\,(t,x) \in \mathcal{D},$$
$$\mathbb{E}\Big[\sup_{t\leqslant s\leqslant\zeta^{t,x}} \big|\nabla_x X_s^{t,x} - I\big|^m\Big] \leqslant \Gamma_m^{(2.4)}(u-t)^{\frac{\alpha m}{2(P+1)}}\,\varrho^{\frac{\alpha m P}{2(P+1)}}. \tag{2.1.18}$$

Proof. Consider $R \leqslant (c_{2m}^{(2.2)} \vee 1)$ as well as $(t,x) \in \mathcal{D}$. We omit to specify the dependence upon (t,x) of $\zeta^{t,x}$ and $X^{t,x}$.

From (2.0.7), we know that there exists a constant $C_m^{(2.5)}$, only depending on k, λ, m and P, such that:

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\leqslant s\leqslant\zeta}|\nabla_x X_s - I|^m\Big]\\
&\quad\leqslant C_m^{(2.5)}\,\mathbb{E}\Big[\sup_{t\leqslant s\leqslant\zeta}|\nabla_x X_s|^m\Big(\int_t^\zeta\big(1+|Z_r|^2\big)\,\mathrm{d}r\Big)^{m/2}\Big] \qquad (2.1.19)\\
&\quad\leqslant C_m^{(2.5)}\,\mathbb{E}\Big[\sup_{t\leqslant s\leqslant\zeta}|\nabla_x X_s|^{2m}\Big]^{1/2}\mathbb{E}\Big[\Big(\int_t^\zeta(1+|Z_r|^2)\,\mathrm{d}r\Big)^m\Big]^{1/2}.
\end{aligned}
$$

Let us now estimate the quantity $\mathbb{E}[(\int_t^\zeta |Z_r|^2\,\mathrm{d}r)^m]$. From Itô's formula, we know that:

$$
\begin{aligned}
\int_t^\zeta |Z_s|^2\,\mathrm{d}s &= |Y_\zeta - Y_t|^2\\
&\quad + 2\int_t^\zeta \langle Y_s - Y_t, e(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s - 2\int_t^\zeta\langle Y_s - Y_t, Z_s\,\mathrm{d}B_s\rangle. \qquad (2.1.20)
\end{aligned}
$$

Therefore, thanks to Theorem 0.1 and to (2.1.5), there exists a constant $C_m^{(2.6)}$ (whose value may change from one inequality to another), only depending on L, λ, Λ, m, P, Q and T, such that:

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_t^\zeta|Z_s|^2\,\mathrm{d}s\Big)^m\Big] &\leqslant C_m^{(2.6)}\Big(\mathbb{E}\Big[\sup_{t\leqslant s\leqslant\zeta}|Y_s - Y_t|^{2m}\Big] + \mathbb{E}\big[(\zeta-t)^m\big]\\
&\quad + R^{\alpha m}\,\mathbb{E}\Big[\Big(\int_t^\zeta |Z_s|^2\,\mathrm{d}s\Big)^m\Big] \qquad (2.1.21)\\
&\quad + \mathbb{E}\Big[\sup_{t\leqslant s\leqslant\zeta}|Y_s - Y_t|^{2m}\Big]^{1/2}\mathbb{E}\Big[\Big(\int_t^\zeta|Z_s|^2\,\mathrm{d}s\Big)^m\Big]^{1/2}\Big).
\end{aligned}
$$

Hence, we have for a small enough R:

$$
\mathbb{E}\Big[\Big(\int_t^\zeta|Z_s|^2\,\mathrm{d}s\Big)^m\Big] \leqslant C_m^{(2.6)}\Big(\mathbb{E}\Big[\sup_{t\leqslant s\leqslant\zeta}|Y_s-Y_t|^{2m}\Big] + \mathbb{E}\big[(\zeta-t)^m\big]\Big). \qquad (2.1.22)
$$

Therefore, we deduce from Theorem 1.3:

$$
\mathbb{E}\Big[\Big(\int_t^\zeta|Z_s|^2\,\mathrm{d}s\Big)^m\Big] \leqslant C_m^{(2.6)}\,\mathbb{E}\big[(\zeta-t)^{\alpha m}\big]. \qquad (2.1.23)
$$

Let us now estimate the quantity $\mathbb{E}[(\zeta-t)^{\alpha m}]$:

Lemma 2.3. *For every $N\in\mathbb{N}^*$, there exists a constant $\Gamma_N^{(2.5)}$, only depending on λ, Λ, N, P, Q and T, such that:*

$$
\forall\, t\leqslant s\leqslant\zeta,\qquad \mathbb{E}\big[(\zeta-s)^N \mid \mathcal{F}_s\big] \leqslant \Gamma_N^{(2.5)}(u-t)^{N/(P+1)}\varrho^{NP/(P+1)}. \qquad (2.1.24)
$$

Proof. From Theorem 4, Paragraph 2, Chapter II of Krylov [13], we know that the property (2.1.24) holds with $N = 1$. Now, note that for every $N \geqslant 1$:

$$\begin{aligned}\mathbb{E}\big[(\zeta - s)^{N+1} \mid \mathcal{F}_s\big] &= (N+1)\,\mathbb{E}\left[\int_s^\zeta (\zeta - r)^N \,\mathrm{d}r \,\middle|\, \mathcal{F}_s\right] \\ &= (N+1)\,\mathbb{E}\left[\int_s^\zeta \mathbb{E}\big[(\zeta - r)^N \mid \mathcal{F}_r\big] \,\mathrm{d}r \,\middle|\, \mathcal{F}_s\right],\end{aligned} \tag{2.1.25}$$

for all $t \leqslant s \leqslant \zeta$. Using an induction, we complete the proof. □

Let us complete the proof of Lemma 2.2. Let us define N as the largest integer less than or equal to αm. From the inequality

$$\mathbb{E}\big[(\zeta - t)^{\alpha m}\big] \leqslant \mathbb{E}\big[(\zeta - t)^{N+1}\big]^{\alpha m/(N+1)}, \tag{2.1.26}$$

(2.1.23) and Lemma 2.3, we deduce the result.

2.2 Properties of the Operator Associated to X

This subsection is devoted to the study of harmonic functions with respect to the operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{P} a_{i,j}\big(\,.\,,\,.\,, \varphi(\,.\,,\,.\,)\big) \frac{\partial^2}{\partial x_i \, \partial x_j},$$

and in particular aims to present the proof of the integration by parts formula that we will use to estimate the gradient of the solution θ. As mentioned in the introduction, this formula is basically due in a preliminary version to Bismut [2] and Elworthy and Li [7] and in its final form to Thalmaier [26]. Hence, our main work is to adapt the proof given in the former article to the case of parabolic operators. Nevertheless, as announced in the introduction, note that we also show how this technique permits to establish the partial differentiability of the function u given in (0.1).

Keeping the notations introduced in (2.0.12) and (2.0.13), we firstly recall the following result (see also Thalmaier [26]):

Proposition 2.3. *Let $(t, x) \in \mathcal{D}$ and ℓ be defined by:*

$$\forall\,(s, y) \in \overline{\mathcal{D}}, \qquad \ell(s, y) = \big(\varrho^2 - |y - z|^2\big)(u - s). \tag{2.2.1}$$

Moreover, set:

$$\forall\, t \leqslant s \leqslant \zeta^{t,x}, \qquad \Sigma^{t,x}(s) = \int_t^s \ell^{-2}\big(r, X_r^{t,x}\big)\,\mathrm{d}r, \tag{2.2.2}$$

then,

$$\Sigma^{t,x}\big(\zeta^{t,x}\big) = +\infty. \tag{2.2.3}$$

Proof. Fix $(t,x) \in \mathcal{D}$. We omit to specify the dependence upon (t,x) of $X^{t,x}$, $\zeta^{t,x}$, $\zeta^{n,t,x}$ and $\Sigma^{t,x}$.

We define:

$$\begin{cases} \forall\, 0 \leqslant S \leqslant \Sigma(\zeta), & \mathcal{T}(S) = \inf\{s \geqslant t,\ \Sigma(s) \geqslant S\}, \\ \forall\, S > \Sigma(\zeta), & \mathcal{T}(S) = \zeta. \end{cases} \tag{2.2.4}$$

Thus, for every nonnegative real S, $\mathcal{T}(S)$ is a stopping time less than ζ.

Let $n \in \mathbb{N}^*$. From Itô's formula, we have for every $S \geqslant 0$:

$$\begin{aligned} &\mathbb{E}\big[\ell^{-1}\big(\mathcal{T}(S) \wedge \zeta^n, X_{\mathcal{T}(S)\wedge\zeta^n}\big)\big] - \ell^{-1}(t,x) \\ &\quad = \mathbb{E}\bigg[\int_t^{\mathcal{T}(S)\wedge\zeta^n} \mathcal{L}\ell^{-1}(r,X_r)\,\mathrm{d}r\bigg] \\ &\quad = \mathbb{E}\bigg[\int_t^{\mathcal{T}(S)} \mathbf{1}_{\{S \leqslant \Sigma(\zeta)\}}\,\mathbf{1}_{[t,\zeta^n]}(r)\mathcal{L}\ell^{-1}(r,X_r)\,\mathrm{d}r\bigg] \\ &\qquad + \mathbb{E}\bigg[\int_t^{\mathcal{T}(S)} \mathbf{1}_{\{S > \Sigma(\zeta)\}}\,\mathbf{1}_{[t,\zeta^n]}(r)\mathcal{L}\ell^{-1}(r,X_r)\,\mathrm{d}r\bigg] \\ &\quad = \mathbb{E}\bigg[\int_0^{S} \mathbf{1}_{\{S \leqslant \Sigma(\zeta)\}}\,\mathbf{1}_{[t,\zeta^n]}\big(\mathcal{T}(r)\big)\big(\ell^2\mathcal{L}\ell^{-1}\big)\big(\mathcal{T}(r),X_{\mathcal{T}(r)}\big)\,\mathrm{d}r\bigg] \\ &\qquad + \mathbb{E}\bigg[\int_0^{\Sigma(\zeta)} \mathbf{1}_{\{S > \Sigma(\zeta)\}}\,\mathbf{1}_{[t,\zeta^n]}\big(\mathcal{T}(r)\big)\big(\ell^2\mathcal{L}\ell^{-1}\big)\big(\mathcal{T}(r),X_{\mathcal{T}(r)}\big)\,\mathrm{d}r\bigg]. \end{aligned} \tag{2.2.5}$$

Noting that there exists a constant $C^{(2.7)}$ (whose value may change from one inequality to another), only depending on Λ and T, such that:

$$\forall\,(s,y) \in \mathcal{D}, \qquad \big(\ell^2\mathcal{L}\ell^{-1}\big)(s,y) \leqslant C^{(2.7)}\ell^{-1}(s,y), \tag{2.2.6}$$

we deduce that: $\forall\, n \in \mathbb{N}^*$, $\forall\, S \geqslant 0$,

$$\begin{aligned} &\mathbb{E}\big[\ell^{-1}\big(\mathcal{T}(S) \wedge \zeta^n, X_{\mathcal{T}(S)\wedge\zeta^n}\big)\big] \\ &\qquad \leqslant \ell^{-1}(t,x) + C^{(2.7)}\int_0^S \mathbb{E}\big[\ell^{-1}\big(\mathcal{T}(r)\wedge\zeta^n, X_{\mathcal{T}(r)\wedge\zeta^n}\big)\big]\,\mathrm{d}r. \end{aligned} \tag{2.2.7}$$

Hence, from Gronwall's lemma, we have: $\forall\, n \in \mathbb{N}^*$, $\forall\, S \geqslant 0$,

$$\mathbb{E}\big[\ell^{-1}\big(\mathcal{T}(S) \wedge \zeta^n, X_{\mathcal{T}(S)\wedge\zeta^n}\big)\big] \leqslant \ell^{-1}(t,x)\exp\big(C^{(2.7)}S\big). \tag{2.2.8}$$

Hence, we deduce: $\forall\, n \in \mathbb{N}^*$, $\forall\, S \geqslant 0$,

$$\begin{aligned} \mathbb{P}\big\{\mathcal{T}(S) \geqslant \zeta^n\big\} &\leqslant \frac{C^{(2.7)}}{n}\,\mathbb{E}\big[\ell^{-1}\big(\mathcal{T}(S) \wedge \zeta^n, X_{\mathcal{T}(S)\wedge\zeta^n}\big)\big] \\ &\leqslant \frac{C^{(2.7)}}{n}\,\ell^{-1}(t,x)\exp\big(C^{(2.7)}S\big). \end{aligned} \tag{2.2.9}$$

From the inclusions $\{\Sigma(\zeta) < S\} \subset \{\mathcal{T}(S) = \zeta\} \subset \{\mathcal{T}(S) \geqslant \zeta^n\}$, we deduce:

$$\forall\, S \geqslant 0, \qquad \mathbb{P}\big\{\Sigma(\zeta) < S\big\} \leqslant \frac{1}{n}\, C^{(2.7)} \ell^{-1}(t,x) \exp\big(C^{(2.7)} S\big). \tag{2.2.10}$$

This completes the proof. □

Notations. From Proposition 2.3, we can define for every $(t,x) \in \mathcal{D}$:

$$\forall\, S \geqslant 0, \qquad \mathcal{T}^{t,x}(S) = \inf\{s \geqslant t,\ \Sigma^{t,x}(s) \geqslant S\}. \tag{2.2.11}$$

Proposition 2.4. *Let $(t,x) \in \mathcal{D}$, $(c,S) \in (\mathbb{R}_+^*)^2$ and $\nu \in \mathbb{R}^P$. Then, setting:*

$$\forall\, s \geqslant t, \qquad h_0(s) = \int_t^{s \wedge \mathcal{T}^{t,x}(S)} \ell^{-2}\big(r, X_r^{t,x}\big)\, \mathrm{d}r, \tag{2.2.12}$$

as well as:

$$\forall\, s \geqslant 0, \qquad h_1(s) = \frac{c}{1 - \mathrm{e}^{-cS}} \int_0^s \mathrm{e}^{-cr}\, \mathrm{d}r = \frac{1}{1 - \mathrm{e}^{-cS}} (1 - \mathrm{e}^{-cs}), \tag{2.2.13}$$

the process h given by:

$$\forall\, s \geqslant t, \qquad h(s) = h_1\big(h_0(s)\big)\nu,$$

is absolutely continuous and satisfies $h(t) = 0$ and $h(s) = \nu$ for $s \geqslant \mathcal{T}^{t,x}(S)$.

Moreover, for every $m \geqslant 1$:

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_t^{\zeta^{t,x}} |\dot{h}_r|^2\, \mathrm{d}r\bigg)^m\bigg] &\leqslant (u-t)^{m-1} |\nu|^{2m} \\ &\quad \times \Big(\frac{c}{1 - \mathrm{e}^{-cS}}\Big)^{2m} \frac{1 - \mathrm{e}^{-C_m^{(2.9)} S}}{C_m^{(2.9)}}\, \ell^{-4m+2}(t,x).\end{aligned} \tag{2.2.14}$$

where:

$$C_m^{(2.9)} = 2cm - C_m^{(2.8)} (u-t) \varrho^2, \tag{2.2.15}$$

and $C_m^{(2.8)}$ is a positive constant only depending on Λ, m and T.

Proof. We keep the notations given in the statement, but as usual we omit to specify the dependence of $X^{t,x}$, $\zeta^{t,x}$, $\zeta^{n,t,x}$ and $\mathcal{T}^{t,x}$ upon (t,x).

Let $m \geqslant 1$. We have:

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_t^{\zeta} |\dot{h}_r|^2\, \mathrm{d}r\bigg)^m\bigg] &\leqslant (u-t)^{m-1}\, \mathbb{E}\bigg[\int_t^{\zeta} |\dot{h}_r|^{2m}\, \mathrm{d}r\bigg] \\ &\leqslant (u-t)^{m-1} |\nu|^{2m} \Big(\frac{c}{1 - \mathrm{e}^{-cS}}\Big)^{2m}\, \mathbb{E}\bigg[\int_t^{\mathcal{T}(S)} \mathrm{e}^{-2cm h_0(r)} \ell^{-4m}(r, X_r)\, \mathrm{d}r\bigg] \\ &= (u-t)^{m-1} |\nu|^{2m} \Big(\frac{c}{1 - \mathrm{e}^{-cS}}\Big)^{2m}\, \mathbb{E}\bigg[\int_0^{S} \mathrm{e}^{-2cmr} \ell^{-4m+2}\big(\mathcal{T}(r), X_{\mathcal{T}(r)}\big)\, \mathrm{d}r\bigg] \\ &= (u-t)^{m-1} |\nu|^{2m} \Big(\frac{c}{1 - \mathrm{e}^{-cS}}\Big)^{2m} \int_0^{S} \mathrm{e}^{-2cmr}\, \mathbb{E}\big[\ell^{-4m+2}\big(\mathcal{T}(r), X_{\mathcal{T}(r)}\big)\big]\, \mathrm{d}r.\end{aligned} \tag{2.2.16}$$

Let us now estimate the term $\mathbb{E}[\ell^{-4m+2}(\mathcal{T}(r), X_{\mathcal{T}(r)})]$ in (2.2.16).

Let us recall that, for all $n \in \mathbb{N}^*$,

$$\zeta^n = \inf\Big\{s \geqslant t,\ |X_s - z| \geqslant \varrho\Big(1 - \frac{1}{n}\Big)\Big\} \wedge \Big(u_0 + (u - u_0)\Big(1 - \frac{1}{n}\Big)\Big). \quad (2.2.17)$$

Then, from Itô's formula, we have for $n \in \mathbb{N}^*$ and $r \leqslant S$,

$$\begin{aligned}
&\mathbb{E}\big[\ell^{-4m+2}\big(\mathcal{T}(r) \wedge \zeta^n, X_{\mathcal{T}(r)\wedge\zeta^n}\big)\big] - \ell^{-4m+2}(t,x) \\
&\quad = \mathbb{E}\Big[\int_t^{\mathcal{T}(r)\wedge\zeta^n} \mathcal{L}\ell^{-4m+2}(s, X_s)\,\mathrm{d}s\Big] \\
&\quad = \mathbb{E}\Big[\int_t^{\mathcal{T}(r)} \mathbf{1}_{[t,\zeta^n]}(s)\mathcal{L}\ell^{-4m+2}(s, X_s)\,\mathrm{d}s\Big] \\
&\quad = \mathbb{E}\Big[\int_0^r \mathbf{1}_{[t,\zeta^n]}\big(\mathcal{T}(s)\big)\big(\ell^2\mathcal{L}\ell^{-4m+2}\big)\big(\mathcal{T}(s), X_{\mathcal{T}(s)}\big)\,\mathrm{d}s\Big].
\end{aligned} \quad (2.2.18)$$

Noting that there exists a positive constant $C_m^{(2.8)}$, only depending on Λ, m and T, such that: $\forall\,(s,y) \in \mathcal{D}$, $s \geqslant t$,

$$\big(\ell^2\mathcal{L}\ell^{-4m+2}\big)(s,y) \leqslant C_m^{(2.8)}(u-t)\varrho^2\ell^{-4m+2}(s,y), \quad (2.2.19)$$

we have:

$$\begin{aligned}
&\mathbb{E}\big[\ell^{-4m+2}\big(\mathcal{T}(r) \wedge \zeta^n, X_{\mathcal{T}(r)\wedge\zeta^n}\big)\big] \leqslant \ell^{-4m+2}(t,x) \\
&\qquad + C_m^{(2.8)}(u-t)\varrho^2 \int_0^r \mathbb{E}\big[\ell^{-4m+2}\big(\mathcal{T}(s) \wedge \zeta^n, X_{\mathcal{T}(s)\wedge\zeta^n}\big)\big]\,\mathrm{d}s.
\end{aligned} \quad (2.2.20)$$

Using Gronwall's lemma and letting $n \to +\infty$, we deduce that for $r \leqslant S$:

$$\mathbb{E}\big[\ell^{-4m+2}\big(\mathcal{T}(r), X_{\mathcal{T}(r)}\big)\big] \leqslant \ell^{-4m+2}(t,x)\exp\big(C_m^{(2.8)}(u-t)\varrho^2 r\big). \quad (2.2.21)$$

Hence, from (2.2.16) and (2.2.21),

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^\zeta |\dot{h}_r|^2\,\mathrm{d}r\Big)^m\Big] \leqslant (u-t)^{m-1}|\nu|^{2m} \\
&\qquad \times \Big(\frac{c}{1 - \mathrm{e}^{-cS}}\Big)^{2m} \int_0^S \mathrm{e}^{-(2cm - C_m^{(2.8)}(u-t)\varrho^2)r}\,\ell^{-4m+2}(t,x)\,\mathrm{d}r.
\end{aligned} \quad (2.2.22)$$

Set:

$$C_m^{(2.9)} = 2cm - C_m^{(2.8)}(u-t)\varrho^2. \quad (2.2.23)$$

Hence,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^\zeta |\dot{h}_r|^2\,\mathrm{d}r\Big)^m\Big] \leqslant (u-t)^{m-1}|\nu|^{2m} \\
&\qquad \times \Big(\frac{c}{1 - \mathrm{e}^{-cS}}\Big)^{2m} \frac{1 - \mathrm{e}^{-C_m^{(2.9)}S}}{C_m^{(2.9)}}\,\ell^{-4m+2}(t,x). \qquad \square
\end{aligned} \quad (2.2.24)$$

Notations. For every $(t,x)\in\mathcal{D}$, we define $n_0^{t,x}$ as the smallest positive integer such that:

$$(t,x)\in\mathcal{D}_{n_0^{t,x}}. \tag{2.2.25}$$

Here is the main result of this subsection:

Theorem 2.1. *Let $w:\overline{\mathcal{D}}\setminus\mathcal{D}\to\mathbb{R}$ be bounded and measurable and v be the function given by:*

$$\forall\,(t,x)\in\overline{\mathcal{D}},\qquad v(t,x)=\mathbb{E}\big[w\big(\zeta^{t,x},X^{t,x}_{\zeta^{t,x}}\big)\big]. \tag{2.2.26}$$

Then, for every $(t,x)\in\mathcal{D}$, for every $i\in\{1,\dots,P\}$, the partial derivative $\partial v/\partial x_i(t,x)$ exists and is given by:

$$\begin{aligned}\frac{\partial v}{\partial x_i}(t,x)=-\mathbb{E}\Bigg[&w\big(\zeta^{t,x},X_{\zeta^{t,x}}\big)\\ &\times\int_t^{\zeta^{t,x}}\big\langle\sigma^{-1}\big(r,X_r^{t,x},Y_r^{t,x}\big)\nabla_x X_r^{t,x}\dot{h}_r^i,\mathrm{d}B_r\big\rangle\Bigg],\end{aligned} \tag{2.2.27}$$

where, for every $i\in\{1,\dots,P\}$, h^i is an $\mathbb{R}^P$-valued bounded adapted process satisfying:

$$\begin{cases} s\in[t,T]\mapsto h_s^i \text{ is absolutely continuous,}\\ \exists\,\eta>0 \text{ such that } \mathbb{E}\Big[\Big(\int_t^T|\dot{h}_s^i|^2\,\mathrm{d}s\Big)^{(1+\eta)/2}\Big]<\infty,\end{cases} \tag{2.2.28}$$

and the boundary conditions:

$$\begin{cases}\exists\,n\in\mathbb{N}^*,\ n\geqslant n_0^{t,x},\ \forall\,s\geqslant\zeta^n, \qquad h_s^i=0,\\ h_t^i=e_i.\end{cases} \tag{2.2.29}$$

Note from Proposition 2.4 that such a process does exist.

Proof. Fix $(t,x)\in\mathcal{D}$ and $i\in\{1,\dots,P\}$. As usual, we omit to specify the dependence upon (t,x) of $X^{t,x}$, $\zeta^{t,x}$, $\zeta^{n,t,x}$ and $n_0^{t,x}$.

Let $n\geqslant n_0$ be an arbitrarily fixed integer and let $\varepsilon_0>0$ be a real such that:

$$\{t\}\times\overline{B}_P(x,\varepsilon_0)\subset\mathcal{D}. \tag{2.2.30}$$

Consider h satisfying (2.2.28) and (2.2.29) with respect to i and n.

We define for every $\varepsilon\in\mathbb{R}$:

$$\forall\,t\leqslant s\leqslant T,\qquad X_s^\varepsilon=X_s^{t,x+\varepsilon h_s}. \tag{2.2.31}$$

Hence, thanks to the boundedness of h and to (2.0.8), we deduce that there exists a constant $C^{(2.10)}$, such that:

$$\forall-\varepsilon_0\leqslant\varepsilon\leqslant\varepsilon_0,\qquad \mathbb{E}\Big[\sup_{t\leqslant s\leqslant T}|X_s^\varepsilon-X_s|^2\Big]\leqslant C^{(2.10)}\varepsilon^2. \tag{2.2.32}$$

Moreover, from Theorem 3.3.1 of Chapter III of Kunita [17], we know that, for every $\varepsilon \in \mathbb{R}$, X^ε is a semimartingale given by:

$$\begin{aligned} \mathrm{d}X_s^\varepsilon &= \sigma\big(s, X_s^\varepsilon, \varphi(s, X_s^\varepsilon)\big)\,\mathrm{d}B_s + \varepsilon\,\nabla_x X_s^{t,x+\varepsilon h_s}\dot{h}_s\,\mathrm{d}s \\ &= \sigma\big(s, X_s^\varepsilon, \varphi(s, X_s^\varepsilon)\big)\Big(\mathrm{d}B_s + \varepsilon\sigma^{-1}\big(s, X_s^\varepsilon, \varphi(s, X_s^\varepsilon)\big)\nabla_x X_s^{t,x+\varepsilon h_s}\dot{h}_s\,\mathrm{d}s\Big). \end{aligned} \tag{2.2.33}$$

Let us then define the following stopping times:

$$\begin{cases} \tau^\varepsilon = \inf\{s \geqslant t,\ (s, X_s^\varepsilon) \notin \mathcal{D}\}, \\ \varrho^\varepsilon = \inf\{s \geqslant t,\ \varepsilon\big|\int_t^s \big\langle \sigma^{-1}\big(r, X_r^\varepsilon, \varphi(r, X_r^\varepsilon)\big)\nabla_x X_r^{t,x+\varepsilon h_r}\dot{h}_r, \mathrm{d}B_r\big\rangle\big| \geqslant 1\big\}, \\ \sigma^{n,\varepsilon} = \zeta^n \wedge \tau^\varepsilon \wedge \varrho^\varepsilon. \end{cases} \tag{2.2.34}$$

Finally, we define the following process:

$$B_s^\varepsilon = B_s + \varepsilon \int_t^s \sigma^{-1}\big(r, X_r^\varepsilon, \varphi(r, X_r^\varepsilon)\big)\nabla_x X_r^{t,x+\varepsilon h_r}\dot{h}_r\,\mathrm{d}r. \tag{2.2.35}$$

and

$$\begin{aligned} G_s^\varepsilon = \exp\Big(&-\varepsilon \int_t^s \langle \sigma^{-1}\big(r, X_r^\varepsilon, \varphi(r, X_r^\varepsilon)\big)\nabla_x X_r^{t,x+\varepsilon h_r}\dot{h}_r, \mathrm{d}B_r\rangle \\ &- \frac{\varepsilon^2}{2}\int_t^s |\sigma^{-1}\big(r, X_r^\varepsilon, \varphi(r, X_r^\varepsilon)\big)\nabla_x X_r^{t,x+\varepsilon h_r}\dot{h}_r|^2\,\mathrm{d}r\Big). \end{aligned} \tag{2.2.36}$$

Fix $-\varepsilon_0 < \varepsilon < \varepsilon_0$.

Applying the Girsanov theorem, we deduce from (2.2.33) and from the pathwise uniqueness of (2.0.2) that:

$$v(t, x + \varepsilon e_i) = \mathbb{E}\big[v\big(\sigma^{n,\varepsilon}, X_{\sigma^{n,\varepsilon}}^\varepsilon\big)G_{\sigma^{n,\varepsilon}}^\varepsilon\big]. \tag{2.2.37}$$

Hence,

$$\begin{aligned} v(t, x + \varepsilon e_i) &= \mathbb{E}\big[v\big(\zeta^n, X_{\zeta^n}^\varepsilon\big)G_{\zeta^n}^\varepsilon\,\mathbf{1}_{\{\zeta^n \leqslant \tau^\varepsilon\}}\,\mathbf{1}_{\{\varrho^\varepsilon = T\}}\big] \\ &\quad + \mathbb{E}\big[v\big(\sigma^{n,\varepsilon}, X_{\sigma^{n,\varepsilon}}^\varepsilon)G_{\sigma^{n,\varepsilon}}^\varepsilon\,\mathbf{1}_{(\{\zeta^n \leqslant \tau^\varepsilon\}\cap\{\varrho^\varepsilon = T\})^c}\big]. \end{aligned} \tag{2.2.38}$$

Since $h(s) = 0$ for $s \geqslant \zeta_n$, note from (2.2.31) that, for every $\varepsilon > 0$, $X_{\zeta^n}^\varepsilon = X_{\zeta^n}$. Therefore:

$$\begin{aligned} v(t, x + \varepsilon e_i) &= \mathbb{E}\big[v\big(\zeta^n, X_{\zeta^n}\big)G_{\zeta^n}^\varepsilon\,\mathbf{1}_{\{\zeta^n \leqslant \tau^\varepsilon\}}\,\mathbf{1}_{\{\varrho^\varepsilon = T\}}\big] \\ &\quad + \mathbb{E}\big[v\big(\sigma^{n,\varepsilon}, X_{\sigma^{n,\varepsilon}}^\varepsilon\big)G_{\sigma^{n,\varepsilon}}^\varepsilon\,\mathbf{1}_{(\{\zeta^n \leqslant \tau^\varepsilon\}\cap\{\varrho^\varepsilon = T\})^c}\big]. \end{aligned} \tag{2.2.39}$$

Consider a regular function $\psi : \mathbb{R} \to \mathbb{R}$, equal to the identity on $\{|x| \leqslant 1\}$ and to 0 outside $\{|x| \leqslant 2\}$ and satisfying $|\psi| \leqslant 1$. Hence, setting:

$$G_s^{\varepsilon,\psi} = \exp\Biggl(-\psi\biggl(\varepsilon\int_t^s \bigl\langle \sigma^{-1}\bigl(r, X_r^\varepsilon, \varphi(r, X_r^\varepsilon)\bigr)\nabla_x X_r^{t,x+\varepsilon h_r}\dot{h}_r, \mathrm{d}B_r\bigr\rangle\biggr) - \frac{\varepsilon^2}{2}\int_t^s \bigl|\sigma^{-1}\bigl(r, X_r^\varepsilon, \varphi(r, X_r^\varepsilon)\bigr)\nabla_x X_r^{t,x+\varepsilon h_r}\dot{h}_r\bigr|^2\,\mathrm{d}r\Biggr), \quad (2.2.40)$$

we have:

$$\begin{aligned} v(t, x+\varepsilon e_i) &= \mathbb{E}\bigl[v\bigl(\zeta^n, X_{\zeta^n}\bigr)G_{\zeta^n}^{\varepsilon,\psi}\,\mathbf{1}_{\{\zeta^n\leqslant\tau^\varepsilon\}}\,\mathbf{1}_{\{\varrho^\varepsilon=T\}}\bigr] \\ &\quad + \mathbb{E}\bigl[v\bigl(\sigma^{n,\varepsilon}, X_{\sigma^{n,\varepsilon}}^\varepsilon\bigr)G_{\sigma^{n,\varepsilon}}^\varepsilon\mathbf{1}_{(\{\zeta^n\leqslant\tau^\varepsilon\}\cap\{\varrho^\varepsilon=T\})^c}\bigr]. \\ &= \mathbb{E}\bigl[v\bigl(\zeta^n, X_{\zeta^n}\bigr)G_{\zeta^n}^{\varepsilon,\psi}\bigr] - \mathbb{E}\bigl[v\bigl(\zeta^n, X_{\zeta^n}\bigr)G_{\zeta^n}^{\varepsilon,\psi}\,\mathbf{1}_{(\{\zeta^n\leqslant\tau^\varepsilon\}\cap\{\varrho^\varepsilon=T\})^c}\bigr] \\ &\quad + \mathbb{E}\bigl[v\bigl(\sigma^{n,\varepsilon}, X_{\sigma^{n,\varepsilon}}^\varepsilon\bigr)G_{\sigma^{n,\varepsilon}}^\varepsilon\mathbf{1}_{(\{\zeta^n\leqslant\tau^\varepsilon\}\cap\{\varrho^\varepsilon=T\})^c}\bigr]. \end{aligned} \quad (2.2.41)$$

Hence, putting:

$$R_\varepsilon = -\mathbb{E}\bigl[v\bigl(\zeta^n, X_{\zeta^n}\bigr)G_{\zeta^n}^{\varepsilon,\psi}\,\mathbf{1}_{(\{\zeta^n\leqslant\tau^\varepsilon\}\cap\{\varrho^\varepsilon=T\})^c}\bigr] + \mathbb{E}\bigl[v\bigl(\sigma^{n,\varepsilon}, X_{\sigma^{n,\varepsilon}}^\varepsilon\bigr)G_{\sigma^{n,\varepsilon}}^\varepsilon\mathbf{1}_{(\{\zeta^n\leqslant\tau^\varepsilon\}\cap\{\varrho^\varepsilon=T\})^c}\bigr], \quad (2.2.42)$$

we have:

$$\begin{aligned} \bigl|R_\varepsilon\bigr| &\leqslant 2\exp(1)\,\|v\|_\infty\bigl(\mathbb{P}\{\zeta^n > \tau^\varepsilon\} + \mathbb{P}\{\varrho^\varepsilon < T\}\bigr) \\ &\leqslant 2\exp(1)\,\|w\|_\infty\Biggl(\mathbb{P}\biggl\{\sup_{t\leqslant s\leqslant T}\bigl|X_s^\varepsilon - X_s\bigr| \geqslant \frac{\varrho}{n}\biggr\} \\ &\quad + \mathbb{P}\biggl\{\varepsilon\sup_{t\leqslant s\leqslant T}\biggl|\int_t^s \langle\sigma_r^{-1}\nabla_x X_r^{t,x+\varepsilon h_r}\dot{h}_r, \mathrm{d}B_r\rangle\biggr| \geqslant 1\biggr\}\Biggr), \end{aligned} \quad (2.2.43)$$

where we have noted σ_r^{-1} instead of $\sigma^{-1}(r, X_r^\varepsilon, \varphi(r, X_r^\varepsilon))$.

Thanks to (2.0.8) and to (2.2.32), we deduce that there exists a constant $C^{(2.11)}$, not depending on ε, such that:

$$\bigl|R_\varepsilon\bigr| \leqslant \|w\|_\infty C^{(2.11)}\bigl(n^2\varepsilon^2 + \varepsilon^{1+\eta/2}\bigr). \quad (2.2.44)$$

Moreover, let us recall that:

$$v(t,x) = \mathbb{E}\bigl[v\bigl(\zeta^n, X_{\zeta^n}\bigr)\bigr]. \quad (2.2.45)$$

Hence, we deduce that, for all $\varepsilon \in\,]-\varepsilon_0, \varepsilon_0[$,

$$\frac{1}{\varepsilon}\bigl(v(t, x+\varepsilon e_i) - v(t,x)\bigr) = \mathbb{E}\biggl[v(\zeta^n, X_{\zeta^n})\biggl(\frac{G_{\zeta^n}^{\varepsilon,\psi} - 1}{\varepsilon}\biggr)\biggr] + \frac{1}{\varepsilon}R^\varepsilon. \quad (2.2.46)$$

Letting $\varepsilon \to 0$, we deduce from (2.0.8) that:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(v(t,x+\varepsilon e_i)-v(t,x)\big)$$
$$= -\mathbb{E}\Big[v(\zeta^n, X_{\zeta^n})\int_t^{\zeta^n}\big\langle \sigma^{-1}\big(r,X_r,\varphi(r,X_r)\big)\nabla_x X_r \dot{h}_r, \mathrm{d}B_r\big\rangle\Big]. \quad (2.2.47)$$

Hence, for every $1 \leqslant i \leqslant P$, $\partial v/\partial x_i(t,x)$ exists and is given by:

$$\frac{\partial v}{\partial x_i}(t,x) = -\mathbb{E}\Big[v(\zeta^n, X_{\zeta^n})\int_t^{\zeta^n}\big\langle \sigma^{-1}\big(r,X_r,\varphi(r,X_r)\big)\nabla_x X_r \dot{h}_r, \mathrm{d}B_r\big\rangle\Big]. \quad (2.2.48)$$

Hence, from the martingale property of $\big(v(s,X_s)\big)_{t\leqslant s\leqslant \zeta}$ and the definition of h, we deduce:

$$\frac{\partial v}{\partial x_i}(t,x) = -\mathbb{E}\Big[w(\zeta, X_{\zeta})\int_t^{\zeta}\langle \sigma^{-1}\big(r,X_r,\varphi(r,X_r)\big)\nabla_x X_r \dot{h}_r, \mathrm{d}B_r\rangle\Big], \quad (2.2.49)$$

This completes the proof. □

Actually, the former scheme also permits to deduce estimates of the partial derivatives $(\partial v/\partial x_i)_{1\leqslant i\leqslant P}$. Indeed, keeping the notations introduced in (2.0.12) and (2.0.13), we state:

Theorem 2.2. *Assume that the assumptions of Theorem 2.1 are in force. Then, for all $p \geqslant 2$ and $0<\varepsilon<1$, there exist two constants $c_{p,\varepsilon}^{(2.4)}$ and $\Gamma_{p,\varepsilon}^{(2.6)}$, only depending on ε, k, L, λ, Λ, p, P, Q and T, such that for every $R \leqslant c_{p,\varepsilon}^{(2.4)}$, for every $(t,x)\in\mathcal{D}$ and for every $i\in\{1,\dots,P\}$:*

$$\begin{aligned}\Big|\frac{\partial v}{\partial x_i}(t,x)\Big| &\leqslant \Gamma_{p,\varepsilon}^{(2.6)}\,\mathbb{E}\big[|w|^{p/(p-1)}\big(\zeta,X_\zeta\big)\big]^{(p-1)/p}(u-t)^{-1/2}\\ &\quad\times\Big(\varrho^{2-2/p}\big(\varrho^2-|x-z|^2\big)^{-(2-2/p)}\\ &\quad+\varrho^{2-\frac{2}{(1+\varepsilon)p}+\frac{\alpha P}{2(P+1)}}\big(\varrho^2-|x-z|^2\big)^{-(2-\frac{2}{(1+\varepsilon)p})}\Big).\end{aligned} \quad (2.2.50)$$

Proof. Fix $(t,x)\in\mathcal{D}$ and $i\in\{1,\dots,P\}$. Once again, we omit to specify the dependence upon (t,x) of $X^{t,x}$, $\nabla_x X^{t,x}$ and $\zeta^{t,x}$.

Let $n\geqslant n_0$ and $(c,S)\in(\mathbb{R}_+^*)^2$. Setting:

$$\forall\,(s,y)\in\mathcal{D}_n, \qquad \ell_n(s,y) = \big(\varrho_n^2-|y-z|^2\big)(u_n-s), \quad (2.2.51)$$

with $u_n = u_0+(1-1/n)(u-u_0)$ and $\varrho_n=(1-1/n)\varrho$,

we denote by h the process associated by Proposition 2.4 to the cylinder $\mathcal{D}_n$, the function ℓ_n, the vector e_i and the reals c and S.

It is readily seen that e_i-h sastifies (2.2.28) and (2.2.29) with respect to n.

Hence, from Theorem 2.1, we have to estimate the quantity:

$$\mathbb{E}\Big[w(\zeta, X_\zeta) \int_t^\zeta \langle \sigma^{-1}(r, X_r, Y_r)\, \nabla_x X_r \dot{h}_r, \mathrm{d}B_r \rangle\Big]. \tag{2.2.52}$$

Let $p \geqslant 2$ and $q \in [1,2]$ such that $1/p + 1/q = 1$. We have:

$$\begin{aligned} &\Big|\mathbb{E}\Big[w(\zeta, X_\zeta) \int_t^\zeta \langle \sigma^{-1}(r, X_r, Y_r)\, \nabla_x X_r \dot{h}_r, \mathrm{d}B_r \rangle\Big]\Big| \\ &\quad \leqslant \mathbb{E}\big[|w|^q\big(\zeta, X_\zeta\big)\big]^{1/q}\, \mathbb{E}\Big[\Big|\int_t^\zeta \langle \sigma^{-1}(r, X_r, Y_r)\, \nabla_x X_r \dot{h}_r, \mathrm{d}B_r \rangle\Big|^p\Big]^{1/p}. \end{aligned} \tag{2.2.53}$$

Let us deal with the term $\mathbb{E}[|\int_t^\zeta \langle \sigma^{-1}(r, X_r, Y_r)\, \nabla_x X_r \dot{h}_r, \mathrm{d}B_r \rangle|^p]$ in (2.2.53).

There exists a constant $C_p^{(2.12)}$, only depending on λ and p, such that:

$$\begin{aligned} &\mathbb{E}\Big[\Big|\int_t^\zeta \langle \sigma^{-1}(r, X_r, Y_r)\, \nabla_x X_r \dot{h}_r, \mathrm{d}B_r \rangle\Big|^p\Big] \\ &\quad \leqslant C_p^{(2.12)}\, \mathbb{E}\Big[\Big(\int_t^\zeta |\sigma^{-1}(r, X_r, Y_r)|^2\, |\nabla_x X_r|^2 |\dot{h}_r|^2 \, \mathrm{d}r\Big)^{p/2}\Big] \\ &\quad \leqslant C_p^{(2.12)}\, \mathbb{E}\Big[\sup_{t \leqslant r \leqslant \zeta} |\nabla_x X_r|^p \Big(\int_t^\zeta |\dot{h}_r|^2 \, \mathrm{d}r\Big)^{p/2}\Big]. \end{aligned} \tag{2.2.54}$$

Hence, there exists a constant $C_p^{(2.13)}$, only depending on λ, P and p, such that:

$$\begin{aligned} &\mathbb{E}\Big[\Big|\int_t^\zeta \langle \sigma^{-1}(r, X_r, Y_r)\, \nabla_x X_r \dot{h}_r, \mathrm{d}B_r \rangle\Big|^p\Big] \\ &\quad \leqslant C_p^{(2.13)} \Big(\mathbb{E}\Big[\Big(\int_t^\zeta |\dot{h}_r|^2 \, \mathrm{d}r\Big)^{p/2}\Big] \\ &\qquad + \mathbb{E}\Big[\sup_{t \leqslant r \leqslant \zeta} |\nabla_x X_r - I|^p \Big(\int_t^\zeta |\dot{h}_r|^2 \, \mathrm{d}r\Big)^{p/2}\Big]\Big) \\ &\quad \leqslant C_p^{(2.13)} \Big(\mathbb{E}\Big[\Big(\int_t^\zeta |\dot{h}_r|^2 \, \mathrm{d}r\Big)^{p/2}\Big] \\ &\qquad + \mathbb{E}\Big[\sup_{t \leqslant r \leqslant \zeta} |\nabla_x X_r - I|^{p(1+\varepsilon)/\varepsilon}\Big]^{\varepsilon/(1+\varepsilon)} \\ &\qquad \times \mathbb{E}\Big[\Big(\int_t^\zeta |\dot{h}_r|^2 \, \mathrm{d}r\Big)^{p(1+\varepsilon)/2}\Big]^{1/(1+\varepsilon)}\Big), \end{aligned} \tag{2.2.55}$$

where $0 < \varepsilon < 1$.

Therefore, applying Lemma 2.2 and Proposition 2.4 to the cylinder $\mathcal{D}_n$, we have for $R \leqslant c_{p(1+\varepsilon)/\varepsilon}^{(2.3)}$,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^{\zeta}\langle\sigma^{-1}(r,X_r,Y_r)\,\nabla_x X_r \dot{h}_r,\mathrm{d}B_r\rangle\Big)^p\Big]^{1/p}\\
&\quad\leqslant \big(C_p^{(2.13)}\big)^{\frac{1}{p}}\Bigg((u_n-t)^{\frac{1}{2}-\frac{1}{p}}\frac{c}{1-\mathrm{e}^{-cS}}\\
&\qquad\times\left(\frac{1-\mathrm{e}^{-C^{(2.14)}S}}{C^{(2.14)}}\right)^{\frac{1}{p}}\ell_n^{-(2-\frac{2}{p})}(t,x)\\
&\qquad+\Big(\Gamma^{(2.4)}_{\frac{p(1+\varepsilon)}{\varepsilon}}\Big)^{\frac{\varepsilon}{p(1+\varepsilon)}}(u_n-t)^{\frac{1}{2}-\frac{1}{p(1+\varepsilon)}+\frac{\alpha}{2(P+1)}}\varrho_n^{\frac{\alpha P}{2(P+1)}}\\
&\qquad\times\frac{c}{1-\mathrm{e}^{-cS}}\left(\frac{1-\mathrm{e}^{-C^{(2.15)}S}}{C^{(2.15)}}\right)^{\frac{1}{p(1+\varepsilon)}}\ell_n^{-(2-\frac{2}{p(1+\varepsilon)})}(t,x)\Bigg],
\end{aligned}
\tag{2.2.56}
$$

where:

$$
\begin{cases}
C^{(2.14)}=cp-C^{(2.8)}_{p/2}(u_n-t)\varrho_n^2,\\
C^{(2.15)}=cp(1+\varepsilon)-C^{(2.8)}_{p(1+\varepsilon)/2}(u_n-t)\varrho_n^2.
\end{cases}
\tag{2.2.57}
$$

Choose $c=1/p\times(1+C^{(2.8)}_{p/2}+C^{(2.8)}_{p(1+\varepsilon)/2})(u_n-t)\varrho_n^2$.

Note that such a choice implies $C^{(2.14)}>0$ and $C^{(2.15)}>0$.

Hence, letting $S\to+\infty$, there exists a constant $\Gamma^{(2.6)}_{p,\varepsilon}$, only depending on ε, k, L, λ, Λ, p, P, Q and T, such that:

$$
\begin{aligned}
\left|\frac{\partial v}{\partial x_i}(t,x)\right| &\leqslant \Gamma^{(2.6)}_{p,\varepsilon}\,\mathbb{E}\big[|w|^q\big(\zeta,X_\zeta\big)\big]^{1/q}\\
&\times\Big((u_n-t)^{-1/2}\varrho_n^{2-2/p}\big(\varrho_n^2-|x-z|^2\big)^{-(2-2/p)}\\
&+(u_n-t)^{-1/2}\varrho_n^{2-\frac{2}{(1+\varepsilon)p}+\frac{\alpha P}{2(P+1)}}\big(\varrho_n^2-|x-z|^2\big)^{-(2-\frac{2}{(1+\varepsilon)p})}\Big).
\end{aligned}
\tag{2.2.58}
$$

Letting $n\to+\infty$, we complete the proof. $\square$

The latter result can be slightly simplified:

Theorem 2.3. *Under the assumptions of Theorem 2.1, there exist for every* $p\geqslant 2$ *two constants* $c_p^{(2.5)}$ *and* $\Gamma_p^{(2.7)}$*, only depending on* k, L, λ, Λ, p, P, Q *and* T*, such that for every* $R\leqslant c_p^{(2.5)}$*, for every* $(t,x)\in\mathcal{D}$ *and for every* $i\in\{1,\dots,P\}$*:*

$$
\begin{aligned}
\left|\frac{\partial v}{\partial x_i}(t,x)\right| &\leqslant \Gamma_p^{(2.7)}\,\mathbb{E}\Big[|w|^{p/(p-1)}\big(\zeta^{t,x},X^{t,x}_{\zeta^{t,x}}\big)\Big]^{(p-1)/p}\\
&\qquad\times(u-t)^{-1/2}(\varrho-|x-z|)^{-(2-2/p)}.
\end{aligned}
\tag{2.2.59}
$$

Proof. Consider $(t,x)\in\mathcal{D}$. Let:

$$
\begin{cases}
r=\varrho-|x-z|,\\
\mathcal{D}'=[t,u[\times B(x,r),\\
\zeta'=\inf\{s\geqslant t,\ (s,X_s)\notin\mathcal{D}'\}.
\end{cases}
\tag{2.2.60}
$$

From Theorem 2.2 applied to the cylinder $\mathcal{D}' \subset \mathcal{D}$, we know that that for all $p \geqslant 2$ and $0 < \varepsilon < 1$, the following holds as soon as $R \leqslant c_{p,\varepsilon}^{(2.4)}$:

$$|\nabla_x v(t,x)| \leqslant \Gamma_{p,\varepsilon}^{(2.6)} \, \mathbb{E}\big[|v|^q(\zeta', X_{\zeta'})\big]^{1/q} (u-t)^{-1/2} \\ \times \big(r^{-2+2/p} + r^{-2+2/(p(1+\varepsilon))+\alpha P/(2(P+1))}\big). \quad (2.2.61)$$

Noting that:

$$\mathbb{E}\big[|v|^q(\zeta', X_{\zeta'})\big] = \mathbb{E}\Big[\big|\mathbb{E}\big[w(\zeta, X_\zeta) \mid \mathcal{F}_{\zeta'}\big]\big|^q\Big] \leqslant \mathbb{E}\big[|w|^q(\zeta, X_\zeta)\big], \quad (2.2.62)$$

and choosing $\varepsilon > 0$ satisfying:

$$\frac{2}{(1+\varepsilon)p} + \frac{\alpha P}{2(P+1)} \geqslant \frac{2}{p}, \quad (2.2.63)$$

we complete the proof. □

2.3 Estimate of $\nabla_x \theta$

Here is the final part of our scheme: from Subsections 2.1 and 2.2, we give a global bound of the gradient of the function θ.

Notations. Keep the notations given in (2.0.12) and (2.0.13), and in addition consider a smooth function $g : \overline{\mathcal{D}} \to \mathbb{R}$ such that:

$$\begin{cases} \forall\,(t,x) \in \overline{\mathcal{D}}, & g(t,x) \geqslant 0, \\ \forall\,(t,x) \in \overline{\mathcal{D}} \setminus \mathcal{D}, & g(t,x) = 0, \\ \forall\,(t,x) \in [u_0,u] \times \partial B(z,\varrho), & g'_t(t,x) = g'_x(t,x) = g''_x(t,x) = 0. \end{cases} \quad (2.3.1)$$

Then, from Itô's formula, we have for every $j \in \{1,\ldots,Q\}$ and for every $(t,x) \in \overline{\mathcal{D}}$:

$$(\theta_j g)(t,x) = (\varphi_j g)(t,x) = \mathbb{E}\bigg[\int_t^{\zeta^{t,x}} (\bar{e}_j g)\big(s, X_s^{t,x}\big)\,\mathrm{d}s\bigg] \\ - \mathbb{E}\bigg[\int_t^{\zeta^{t,x}} (\varphi_j \mathcal{L} g)\big(s, X_s^{t,x}\big)\,\mathrm{d}s\bigg] - \mathbb{E}\bigg[\int_t^{\zeta^{t,x}} \langle \nabla_x \varphi_j, \bar{a}\,\nabla_x g\rangle\big(s, X_s^{t,x}\big)\,\mathrm{d}s\bigg]. \quad (2.3.2)$$

where:

$$\forall\,(s,y) \in \overline{\mathcal{D}}, \qquad \begin{cases} \bar{e}(s,y) = e\big(s,y,\varphi(s,y), \nabla_x \varphi(s,y)\,\sigma\big(s,y,\varphi(s,y)\big)\big), \\ \bar{a}(s,y) = a\big(s,y,\varphi(s,y)\big). \end{cases} \quad (2.3.3)$$

Hence, setting for all $(t,x) \in \overline{\mathcal{D}}$, and all $r \in [t,u]$:

$$v^{r,j}(t,x) = \mathbb{E}\big[\big(\bar{e}_j g - \varphi_j \mathcal{L} g - \langle \nabla_x \varphi_j, \bar{a}\,\nabla_x g\rangle\big)\big(r \wedge \zeta^{t,x}, X_{r\wedge\zeta^{t,x}}\big)\big] \\ = \mathbb{E}\big[\mathbf{1}_{[t,\zeta^{t,x}]}(r)\big(\bar{e}_j g - \varphi_j \mathcal{L} g - \langle \nabla_x \varphi_j, \bar{a}\,\nabla_x g\rangle\big)\big(r \wedge \zeta^{t,x}, X_{r\wedge\zeta^{t,x}}\big)\big], \quad (2.3.4)$$

we have:

$$\forall\,(t,x)\in\overline{\mathcal{D}},\qquad (\theta_j g)(t,x)=\int_t^u v^{r,j}(t,x)\,\mathrm{d}r. \tag{2.3.5}$$

Hence, noting that $r\wedge\zeta^{t,x}$ is the first exit time of $X^{t,x}$ from the cylinder $[u_0,r]\times B(z,\varrho)$, we deduce from Theorems 2.1 and 2.3 that for every $i\in\{1,\dots,P\}$:

$$\forall\,(t,x)\in\mathcal{D},\qquad \frac{\partial(\theta_j g)}{\partial x_i}(t,x)=\int_t^u \frac{\partial v^{r,j}}{\partial x_i}(t,x)\,\mathrm{d}r. \tag{2.3.6}$$

Additional notations. For every $n\in\mathbb{N}^*$, we denote by p_n the solution in $]2,+\infty[$ of the equation:

$$p=2+\frac{4}{n}-\frac{4}{pn}. \tag{2.3.7}$$

Actually, p_n is given by:

$$p_n=1+\frac{2}{n}+\sqrt{1+\frac{4}{n^2}}, \tag{2.3.8}$$

which tends to 2 as $n\to+\infty$.

We firstly prove the following local bound of $\nabla_x\theta$:

Theorem 2.4. *Let $n>2$ be such that the following inequality holds:*

$$\frac{p_n}{2}-\beta(p_n-1)<1, \tag{2.3.9}$$

where β is given by Proposition 2.1.

Then, there exists a constant $\Gamma^{(2.8)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that for every $R\leqslant(c_{p_n}^{(2.5)}\wedge c^{(2.1)})$:

$$\sup_{(t,x)\in\mathcal{D}}\Big(|\nabla_x\theta(t,x)|\,(u-t)\,\big(\varrho^2-|x-z|^2\big)^{np_n/2}\Big) \leqslant \Gamma^{(2.8)}(u-u_0)^{1/2}\big(\varrho^{(2p_n-2)/(p_n-2)}+\varrho^{2n-2/p_n}\big). \tag{2.3.10}$$

Proof. Let n satisfy the assumptions of the statement. Put:

$$p=p_n\qquad\text{and}\qquad q=\frac{p}{p-1}<2, \tag{2.3.11}$$

and consider $R\leqslant(c_p^{(2.5)}\wedge c^{(2.1)})$.

Moreover, let:

$$\forall\,(t,x)\in\overline{\mathcal{D}},\qquad g(t,x)=\big(\varrho^2-|x-z|^2\big)^n(u-t). \tag{2.3.12}$$

Hence, there exists a constant $C^{(2.16)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that:

$$\forall\,(t,x)\in\overline{\mathcal{D}},\qquad \begin{cases} |\nabla_x g(t,x)| \leqslant C^{(2.16)}(u-t)\varrho^{2n-1}, \\ |\mathcal{L}g(t,x)| \leqslant C^{(2.16)}\varrho^{2n-2}. \end{cases} \tag{2.3.13}$$

Fix now $(t,x)\in\mathcal{D}$ and $i\in\{1,\dots,P\}$. Once again, we omit to specify the dependence upon (t,x) of $X^{t,x}$ and $\zeta^{t,x}$.

From Theorem 2.3, we know that for every $j\in\{1,\dots,Q\}$,

$$\begin{aligned} \left|\int_t^u \frac{\partial v^{r,j}}{\partial x_i}(t,x)\,\mathrm{d}r\right| &\leqslant \Gamma_p^{(2.7)}(\varrho-|x-z|)^{-(2-2/p)} \\ &\quad\times\int_t^u\Big(\mathbb{E}\big[|\bar{e}_j g|^q(r\wedge\zeta,X_{r\wedge\zeta})\big]^{1/q}+\mathbb{E}\big[|\varphi_j\mathcal{L}g|^q(r\wedge\zeta,X_{r\wedge\zeta})\big]^{1/q} \\ &\qquad+\mathbb{E}\big[|\langle\nabla_x\varphi_j,\bar{a}\,\nabla_x g\rangle|^q(r\wedge\zeta,X_{r\wedge\zeta})\big]^{1/q}\Big)(r-t)^{-1/2}\,\mathrm{d}r. \end{aligned} \tag{2.3.14}$$

Hence, from (2.3.13), there exists a constant $C^{(2.17)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that for every $j\in\{1,\dots,Q\}$,

$$\begin{aligned} \left|\frac{\partial(\theta_j g)}{\partial x_i}(t,x)\right| &\leqslant C^{(2.17)}(\varrho-|x-z|)^{-(2-2/p)} \\ &\quad\times\int_t^u\Big(\mathbb{E}\big[|\bar{e}_j g|^q(r\wedge\zeta,X_{r\wedge\zeta})\big]^{1/q}+\varrho^{2n-2} \\ &\qquad+(u-t)\varrho^{2n-1}\,\mathbb{E}\big[|Z_r|^q\,\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}\Big)(r-t)^{-1/2}\,\mathrm{d}r. \end{aligned} \tag{2.3.15}$$

Therefore, modifying $C^{(2.17)}$ if necessary, we deduce that for every $j\in\{1,\dots,Q\}$,

$$\begin{aligned} &\left|\Big(\frac{\partial\theta_j}{\partial x_i}\,g\Big)(t,x)\right|\big(\varrho-|x-z|\big)^{2-2/p} \\ &\quad\leqslant C^{(2.17)}\bigg((u-t)^{1/2}\varrho^{2n-2}+\int_t^u\Big(\mathbb{E}\big[|\bar{e}_j g|^q(r\wedge\zeta,X_{r\wedge\zeta})\big]^{1/q} \\ &\qquad+(u-t)\varrho^{2n-1}\,\mathbb{E}\big[|Z_r|^q\,\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}\Big)(r-t)^{-1/2}\,\mathrm{d}r\bigg). \end{aligned} \tag{2.3.16}$$

Hence, multiplying (2.3.16) by $(\varrho+|x-z|)^{2-2/p}$, we deduce that there exists a constant $C^{(2.18)}$ (whose value may change from inequality to another), only depending on k, L, λ, Λ, n, P, Q and T, such that for every $j\in\{1,\dots,Q\}$,

$$\begin{aligned} &\big|\nabla_x\theta_j(t,x)\big|\,\big(\varrho^2-|x-z|^2\big)^{n+2-2/p}(u-t)\leqslant C^{(2.18)}\varrho^{2-2/p} \\ &\quad\times\bigg((u-t)^{1/2}\varrho^{2n-2}+\int_t^u\Big(\mathbb{E}\big[|Z_{r\wedge\zeta}|^{2q}g^q(r\wedge\zeta,X_{r\wedge\zeta})\big]^{1/q} \\ &\qquad+(u-t)\varrho^{2n-1}\,\mathbb{E}\big[|Z_r|^q\,\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}\Big)(r-t)^{-1/2}\,\mathrm{d}r\bigg). \end{aligned} \tag{2.3.17}$$

Note that, for all $t\leqslant r\leqslant u$,

$$\begin{aligned}\mathbb{E}\big[|Z_{r\wedge\zeta}|^{2q} g^q(r\wedge\zeta, X_{r\wedge\zeta})\big] &= \mathbb{E}\Big[|Z_{r\wedge\zeta}|^2 \mathbf{1}_{\{r\leqslant\zeta\}} \\ &\times \Big(|Z_{r\wedge\zeta}|\big(\varrho^2 - |X_{r\wedge\zeta} - z|^2\big)^{\frac{np}{2}}(u - r\wedge\zeta)\Big)^{\frac{2}{p-1}}(u - r\wedge\zeta)^{\frac{p-2}{p-1}}\Big].\end{aligned} \tag{2.3.18}$$

Set:

$$\begin{aligned}\overline{M} &= \sup_{(s,y)\in\mathcal{D}} \Big(|\nabla_x\theta(s,y)|\,\big(\varrho^2 - |y-z|^2\big)^{np/2}(u-s)\Big) \\ &= \sup_{(s,y)\in\overline{\mathcal{D}}} \Big(|\nabla_x\theta(s,y)|\,\big(\varrho^2 - |y-z|^2\big)^{np/2}(u-s)\Big) \\ &= \sup_{(s,y)\in\overline{\mathcal{D}}} \Big(|\nabla_x\varphi(s,y)|\,\big(\varrho^2 - |y-z|^2\big)^{np/2}(u-s)\Big).\end{aligned} \tag{2.3.19}$$

From (2.3.17) and (2.3.18) and from the choice of p, we deduce that:

$$\begin{aligned}&\big|\nabla_x\theta(t,x)\big|\,\big(\varrho^2 - |x-z|^2\big)^{np/2}(u-t) \\ &\quad\leqslant C^{(2.18)}\varrho^{2-2/p}\Big((u-t)^{1/2}\varrho^{2n-2} \\ &\qquad + \overline{M}^{2/p}(u-t)^{(p-2)/p}\int_t^u \mathbb{E}\big[|Z_r|^2\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r \\ &\qquad + (u-t)\varrho^{2n-1}\int_t^u \mathbb{E}\big[|Z_r|^q\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r\Big).\end{aligned} \tag{2.3.20}$$

Moreover, from Young's inequality:

$$\begin{aligned}&\int_t^u \mathbb{E}\big[|Z_r|^2\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r \\ &\qquad + \int_t^u \mathbb{E}\big[|Z_r|^q\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r \\ &\quad\leqslant \int_t^u \mathbb{E}\Big[\frac{|Z_r|^2}{(r-t)^\beta}\mathbf{1}_{\{r\leqslant\zeta\}}\Big]^{1/q}(r-t)^{-1/2+\beta/q}\,\mathrm{d}r \\ &\qquad + \int_t^u \mathbb{E}\Big[\frac{|Z_r|^2}{(r-t)^\beta}\mathbf{1}_{\{r\leqslant\zeta\}}\Big]^{1/2}(r-t)^{-(1-\beta)/2}\,\mathrm{d}r \\ &\quad\leqslant \int_t^u \Big(\mathbb{E}\Big[\frac{|Z_r|^2}{(r-t)^\beta}\mathbf{1}_{\{r\leqslant\zeta\}}\Big] + (r-t)^{-p/2+\beta(p-1)} + (r-t)^{-1+\beta}\Big)\,\mathrm{d}r.\end{aligned} \tag{2.3.21}$$

Hence, from the inequalities (2.3.20) and (2.3.21), and thanks to Proposition 2.1 and to the choice of n, there exists a constant $\Gamma^{(2.8)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that:

$$\begin{aligned}&\big|\nabla_x\theta(t,x)\big|\,\big(\varrho^2 - |x-z|^2\big)^{pn/2}(u-t) \\ &\qquad\leqslant \Gamma^{(2.8)}\varrho^{2-2/p}\Big((u-t)^{1/2}\varrho^{2n-2} + \overline{M}^{2/p}(u-t)^{(p-2)/p}\Big).\end{aligned} \tag{2.3.22}$$

Finally, taking the supremum over $\mathcal{D}$, we deduce:

$$\overline{M} \leqslant \Gamma^{(2.8)} \varrho^{2-2/p} \Big((u-u_0)^{1/2} \varrho^{2n-2} + \overline{M}^{2/p} (u-u_0)^{(p-2)/p} \Big). \tag{2.3.24}$$

Hence, modifying if necessary $\Gamma^{(2.8)}$, we deduce:

$$\overline{M} \leqslant \Gamma^{(2.8)} (u-u_0)^{1/2} \Big(\varrho^{(2p-2)/(p-2)} + \varrho^{2n-2/p} \Big). \tag{2.3.25}$$

This completes the proof. □

Following the proof of Theorem 2.3, we deduce:

Theorem 2.5. *Under the assumptions of Theorem 2.4, there exists a constant $\Gamma^{(2.9)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that for every $R \leqslant (c^{(2.5)}_{p_n} \wedge c^{(2.1)})$:*

$$\forall\, (t,x) \in \mathcal{D}, \quad |\nabla_x \theta(t,x)| \leqslant \Gamma^{(2.9)} (\varrho - |x-z|)^{-(n+1+\varepsilon_n)} (u-t)^{-1/2}. \tag{2.3.26}$$

where $\varepsilon_n \to 0$ as $n \to +\infty$.

Keeping the notations introduced in (2.0.12) and (2.0.13), we establish the following estimate which holds as soon as t is closed enough from T:

Theorem 2.6. *Assume that $u = t_0 + R^2 = T$, and that n satisfies the assumptions of Theorem 2.4.*

Then, there exists a constant $\Gamma^{(2.10)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that for every $R \leqslant (c^{(2.5)}_{p_n} \wedge c^{(2.1)})$:

$$\begin{aligned} \sup_{(t,x)\in\mathcal{D}} &\Big(|\nabla_x \theta(t,x)| \, \big(\varrho^2 - |x-z|^2\big)^{np_n/2} \Big) \\ &\leqslant \Gamma^{(2.10)} \Big(\varrho^{(2p_n-2)/(p_n-2)} + \varrho^{2n-2/p_n} \Big). \end{aligned} \tag{2.3.27}$$

Proof. Consider a smooth function $g : \overline{\mathcal{D}} \to \mathbb{R}$ such that:

$$\begin{cases} \forall\, (t,x) \in \overline{\mathcal{D}}, \qquad g(t,x) \geqslant 0, \\ \forall\, (t,x) \in [u_0, T] \times \partial B(z,\varrho), \\ \qquad\qquad g(t,x) = g'_t(t,x) = g'_x(t,x) = g''_x(t,x) = 0. \end{cases} \tag{2.3.28}$$

Then, from Itô's formula, we have for every $j \in \{1,\dots,Q\}$ and for every $(t,x) \in \overline{\mathcal{D}}$:

$$\begin{aligned} (\theta_j g)(t,x) &= (\varphi_j g)(t,x) \\ &= \mathbb{E}\big[(\varphi_j g)\big(\zeta^{t,x}, X^{t,x}_{\zeta^{t,x}}\big)\big] + \mathbb{E}\Big[\int_t^{\zeta^{t,x}} (\bar{e}_j g)\big(s, X^{t,x}_s\big)\,\mathrm{d}s\Big] \\ &\quad - \mathbb{E}\Big[\int_t^{\zeta^{t,x}} (\varphi_j \mathcal{L} g)\big(s, X^{t,x}_s\big)\,\mathrm{d}s\Big] - \mathbb{E}\Big[\int_t^{\zeta^{t,x}} \langle \nabla_x \varphi_j, \bar{a}\, \nabla_x g\rangle \big(s, X^{t,x}_s\big)\,\mathrm{d}s\Big]. \end{aligned} \tag{2.3.29}$$

Hence, letting for every $j \in \{1, \dots, Q\}$ and for every $(t,x) \in \overline{\mathcal{D}}$:

$$v^j(t,x) = \mathbb{E}\big[(\varphi_j g)\big(\zeta^{t,x}, X^{t,x}_{\zeta^{t,x}}\big)\big], \tag{2.3.30}$$

and for every $t \leqslant r \leqslant T$,

$$\begin{aligned} v^{r,j}(t,x) &= \mathbb{E}\Big[\big(\bar{e}_j g - \varphi_j \mathcal{L} g - \langle \nabla_x \varphi_j, \bar{a}\, \nabla_x g\rangle\big)\big(r \wedge \zeta^{t,x}, X^{t,x}_{r\wedge\zeta^{t,x}}\big)\Big], \\ &= \mathbb{E}\Big[\mathbf{1}_{[t,\zeta^{t,x}]}(r)\big(\bar{e}_j g - \varphi_j \mathcal{L} g - \langle \nabla_x \varphi_j, \bar{a}\, \nabla_x g\rangle\big)\big(r \wedge \zeta^{t,x}, X^{t,x}_{r\wedge\zeta^{t,x}}\big)\Big], \end{aligned} \tag{2.3.31}$$

we have for every $i \in \{1, \dots, P\}$ and for every $(t,x) \in \mathcal{D}$:

$$\begin{aligned} (\theta_j g)(t,x) &= v^j(t,x) + \int_t^T v^{r,j}(t,x)\,\mathrm{d}r, \\ \frac{\partial(\theta_j g)}{\partial x_i}(t,x) &= \frac{\partial v^j}{\partial x_i}(t,x) + \int_t^T \frac{\partial v^{r,j}}{\partial x_i}(t,x)\,\mathrm{d}r. \end{aligned} \tag{2.3.32}$$

Let $R \leqslant (c^{(2.5)}_{p_n} \wedge c^{(2.1)})$ and let g be given by:

$$\forall\,(t,x) \in \overline{\mathcal{D}}, \qquad g(t,x) = \big(\varrho^2 - |x-z|^2\big)^n. \tag{2.3.33}$$

Hence, there exists a constant $C^{(2.19)}$, only depending on k, L, λ, $\varLambda$, n, P, Q and T, such that:

$$\forall\,(t,x) \in \overline{\mathcal{D}}, \qquad \begin{cases} |\nabla_x g(t,x)| \leqslant C^{(2.19)} \varrho^{2n-1}, \\ |\mathcal{L} g(t,x)| \leqslant C^{(2.19)} \varrho^{2n-2}. \end{cases} \tag{2.3.34}$$

Fix now $(t,x) \in \mathcal{D}$ and $i \in \{1, \dots, P\}$. As usual, we omit to specify the dependence upon (t,x) of $X^{t,x}$ and $\zeta^{t,x}$.

Let us firstly estimate the quantities $(\partial v^j/\partial x_i(t,x))_{1\leqslant j\leqslant Q}$.

Applying Theorem 2.3 to the function $(s,y) \in \overline{\mathcal{D}} \mapsto v^j(s,y) - (H_j g)(x)$, we deduce that for every $j \in \{1, \dots, Q\}$:

$$\begin{aligned} \left|\frac{\partial v^j}{\partial x_i}(t,x)\right| \leqslant \varGamma^{(2.7)}_p\, \mathbb{E}\Big[\big|(H_j g)(X_\zeta) - (H_j g)(x)\big|^q\Big]^{1/q} & \\ \times (T-t)^{-1/2}(\varrho - |x-z|)^{-(2-2/p)}. & \end{aligned} \tag{2.3.35}$$

Hence, from Assumption (A'), there exists a constant $C^{(2.20)}$, only depending on k, L, λ, $\varLambda$, n, P, Q and T, such that for every $j \in \{1, \dots, Q\}$:

$$\begin{aligned} \left|\frac{\partial v^j}{\partial x_i}(t,x)\right| \leqslant C^{(2.20)} \varrho^{2n-1}\, \mathbb{E}\big[|X_\zeta - x|^q\big]^{1/q} & \\ \times (T-t)^{-1/2}(\varrho - |x-z|)^{-(2-2/p)}. & \end{aligned} \tag{2.3.36}$$

Hence, modifying $C^{(2.20)}$ if necessary, we have:

$$\left|\frac{\partial v^j}{\partial x_i}(t,x)\right| \leqslant C^{(2.20)}\,\varrho^{2n-1}(\varrho - |x-z|)^{-(2-2/p)}. \tag{2.3.37}$$

Hence, following the proof of Theorem 2.4, we deduce from (2.3.15) and (2.3.32) that there exists a constant $C^{(2.21)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that for every $j \in \{1,\dots,Q\}$:

$$\begin{aligned}
&|\nabla_x \theta_j(t,x)|\, g(t,x)\,(\varrho - |x-z|)^{2-2/p} \\
&\quad \leqslant C^{(2.21)}\biggl(\varrho^{2n-1} + \int_t^T \Bigl(\varrho^{2n-2} + \mathbb{E}\bigl[|\bar{e}_j g|^q\bigl(r\wedge\zeta, X_{r\wedge\zeta}\bigr)\bigr]^{1/q} \\
&\qquad\qquad + \varrho^{2n-1}\,\mathbb{E}\bigl[\bigl(|Z_r|\,\mathbf{1}_{\{r\leqslant\zeta\}}\bigr)^q\bigr]^{1/q}\Bigr)(r-t)^{-1/2}\,\mathrm{d}r\biggr).
\end{aligned} \tag{2.3.38}$$

Hence, multiplying by $(\varrho + |x-z|)^{2-2/p}$, we deduce that there exists $C^{(2.22)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that for every $j \in \{1,\dots,Q\}$:

$$\begin{aligned}
&|\nabla_x \theta_j(t,x)|\,\bigl(\varrho^2 - |x-z|^2\bigr)^{n+2-2/p} \\
&\quad \leqslant C^{(2.22)}\varrho^{2-2/p}\biggl(\varrho^{2n-2} + \int_t^T \Bigl(\mathbb{E}\bigl[|Z_r|^{2q} g^q\bigl(r\wedge\zeta, X_{r\wedge\zeta}\bigr)\bigr]^{1/q} \\
&\qquad\qquad + \varrho^{2n-1}\,\mathbb{E}\bigl[\bigl(|Z_r|\,\mathbf{1}_{\{r\leqslant\zeta\}}\bigr)^q\bigr]^{1/q}\Bigr)(r-t)^{-1/2}\,\mathrm{d}r\biggr).
\end{aligned} \tag{2.3.39}$$

Note once again that for every $r \in [t,T]$:

$$\begin{aligned}
&\mathbb{E}\bigl[|Z_r|^{2q} g^q\bigl(r\wedge\zeta, X_{r\wedge\zeta}\bigr)\bigr] \\
&\qquad = \mathbb{E}\Bigl[|Z_r|^2\,\mathbf{1}_{\{r\leqslant\zeta\}}\bigl(|Z_r|\,g^{p/2}\bigl(r\wedge\zeta, X_{r\wedge\zeta}\bigr)\bigr)^{2/(p-1)}\Bigr].
\end{aligned} \tag{2.3.40}$$

Let:

$$\overline{M}' = \sup_{(s,y)\in\mathcal{D}} \bigl(|\nabla_x \theta(s,y)|\,g^{p/2}(s,y)\bigr). \tag{2.3.41}$$

Hence, modifying $C^{(2.22)}$ if necessary, we deduce from the choice of p that:

$$\begin{aligned}
&|\nabla_x \theta(t,x)|\,\bigl(g(t,x)\bigr)^{p/2} \leqslant C^{(2.22)}\varrho^{2-2/p} \\
&\quad \times \biggl(\varrho^{2n-2} + \bigl(\overline{M}'\bigr)^{2/p}\int_t^T \mathbb{E}\bigl[|Z_r|^2\,\mathbf{1}_{\{r\leqslant\zeta\}}\bigr]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r \\
&\qquad\qquad + \varrho^{2n-1}\int_t^T \mathbb{E}\bigl[|Z_r|^q\,\mathbf{1}_{\{r\leqslant\zeta\}}\bigr]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r\biggr).
\end{aligned} \tag{2.3.42}$$

Moreover, from (2.3.21):

$$\begin{aligned}
&\int_t^T \mathbb{E}\big[|Z_r|^2\,\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r\\
&\quad+\int_t^T \mathbb{E}\big[|Z_r|^q\,\mathbf{1}_{\{r\leqslant\zeta\}}\big]^{1/q}(r-t)^{-1/2}\,\mathrm{d}r \qquad (2.3.43)\\
&\leqslant \int_t^T \bigg(\mathbb{E}\bigg[\frac{|Z_r|^2}{(r-t)^\beta}\,\mathbf{1}_{\{r\leqslant\zeta\}}\bigg]+(r-t)^{-p/2+\beta(p-1)}+(r-t)^{-1+\beta}\bigg)\,\mathrm{d}r.
\end{aligned}$$

Hence, from the inequalities (2.3.42) and (2.3.43) and Proposition 2.1, there exists a constant $C^{(2.23)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that, for all $(t,x)\in\mathcal{D}$,

$$|\nabla_x\theta(t,x)|\,\big(g(t,x)\big)^{p/2}\leqslant C^{(2.23)}\varrho^{2-2/p}\Big(\varrho^{2n-2}+\big(\overline{M}'\big)^{2/p}\Big). \qquad (2.3.44)$$

Finally, taking the supremum over $\mathcal{D}$:

$$\overline{M}'\leqslant C^{(2.23)}\big(\varrho^{(2p-2)/(p-2)}+\varrho^{2n-2/p}\big). \qquad (2.3.45)$$

This completes the proof. □

We deduce:

Theorem 2.7. *Under the assumptions and the notations of Theorem 2.6, there exists a constant $\Gamma^{(2.11)}$, only depending on k, L, λ, Λ, n, P, Q and T, such that for every $R\leqslant(c_{p_n}^{(2.5)}\wedge c^{(2.1)})$:*

$$\forall\,(t,x)\in\mathcal{D},\qquad |\nabla_x\theta(t,x)|\leqslant\Gamma^{(2.11)}(\varrho-|x-z|)^{-(n+1+\varepsilon_n)}. \qquad (2.3.46)$$

Remark 2.2. From Theorem 1.2 (local Hölder estimate of θ), we can give the following version of Theorem 2.5:

Theorem 2.8. *Assume that n satisfies the assumptions of Theorem 2.4. Then, for every $0<\delta<T$, there exist two constants $c_{\delta,n}^{(2.6)}$ and $\Gamma_{\delta,n}^{(2.12)}$, only depending on δ, k, λ, Λ, n, P, Q and T, such that for all $(t_0,x_0)\in[0,T[\times\mathbb{R}^P$ and $R>0$, satisfying $R\leqslant c_{\delta,n}^{(2.6)}$ and $t_0+R^2\leqslant T-\delta$, we have:*

$$\forall\,(t,x)\in\mathcal{D}, |\nabla_x\theta(t,x)|\leqslant\Gamma_{\delta,n}^{(2.12)}(u-t)^{-1/2}(\varrho-|x-z|)^{-(n+1+\varepsilon_n)}, \qquad (2.3.47)$$

where ε_n is given by Theorem 2.5.

Finally, from Theorems 2.4 and 2.6, we deduce the following global estimate of the gradient of θ:

Theorem 2.9. *Under Assumption (A'), there exists a constant $\Gamma^{(2.13)}$, only depending on k, L, λ, Λ, P, Q and T, such that:*

$$\forall\,(t,x)\in[0,T]\times\mathbb{R}^P,\qquad |\nabla_x\theta(t,x)|\leqslant\Gamma^{(2.13)}. \qquad (2.3.48)$$

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