Self-similar fragmentations and stable subordinators

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Summary. Let $(Y(t), t \geqslant 0)$ be the fragmentation process introduced by Aldous and Pitman that can be obtained by time-reversing the standard additive coalescent. Let $(\sigma_{1/2}(t), t \geqslant 0)$ be the stable subordinator of index 1/2. Aldous and Pitman showed that the distribution of the sizes of the fragments of $Y(t)$ is the same as the conditional distribution of the jump sizes of $\sigma_{1/2}$ up to time t, given $\sigma_{1/2}(t) = 1$. We show that this is a special property of the stable subordinator of index 1/2, in the sense that if $\alpha \neq 1/2$ and σ_α is the stable subordinator of index α, then there exists no self-similar fragmentation for which the distribution of the sizes of the fragments at time t equals the conditional distribution of the jump sizes of σ_α up to time t, given $\sigma_\alpha(t) = 1$. We also show that a property relating the distribution of a size-biased pick from $Y(t)$ to the distribution of $\sigma_{1/2}(t)$ is similarly particular to the $\alpha = 1/2$ case. However, we show that for each $\alpha \in (0, 1)$, there is a family of self-similar fragmentations whose behavior as $t \downarrow 0$ is related to the stable subordinator of index α in the same way that the behavior of $Y(t)$ as $t \downarrow 0$ is related to the stable subordinator of index 1/2.

Key words: Self-similar fragmentation, stable subordinator, Poisson–Kingman distribution.

1 Introduction

Fragmentation processes describe an object that breaks into smaller pieces in a random way as time moves forward. *Ranked fragmentations* are Markov processes taking their values in the set $\Delta = \{(x_i)_{i=1}^\infty : x_1 \geqslant x_2 \geqslant \cdots \geqslant 0, \sum_{i=1}^\infty x_i \leqslant 1\}$. If $(X(t), t \geqslant 0)$ is a ranked fragmentation, we can regard the terms in the sequence $X(t)$ as being the masses of the components into which the object has fragmented after time t, with the masses being ranked in decreasing order. Alternatively, one can study *partition-valued fragmentations*,

J. Azéma et al. (Eds.): LNM 1832, pp. 333–359, 2003.

which take their values in the set of partitions of $\mathbb{N} = \{1, 2, \dots\}$. If $(\Pi(t), t \geqslant 0)$ is a partition-valued fragmentation and $s < t$, then the partition $\Pi(t)$ is a refinement of the partition $\Pi(s)$.

In recent years, a fragmentation introduced in [4] by Aldous and Pitman, which we call the *Aldous–Pitman fragmentation*, has been studied extensively. Aldous and Pitman first constructed this fragmentation process from the Brownian continuum random tree (CRT) of Aldous (see [1, 2, 3]). The CRT is equipped with a finite "mass measure" concentrated on the leaves of the tree and a σ-finite "length measure" on the skeleton of the tree. When the CRT is cut at various points along the skeleton, the tree is split into components whose masses sum to one. Aldous and Pitman defined a ranked fragmentation process $(Y(t), t \geqslant 0)$ such that $Y(t)$ consists of the ranked sequence of masses of tree components after the CRT has been subjected to a Poisson process of cuts at rate t per unit length.

One can also obtain a partition-valued fragmentation $(\Pi(t), t \geqslant 0)$ by picking leaves $U_1, U_2, \dots$ independently from the mass measure of the CRT, and then declaring i and j to be in the same block of $\Pi(t)$ if and only if the leaves U_i and U_j are in the same tree component at time t. To see how this process is related to $(Y(t), t \geqslant 0)$, we first give a definition. If $B \subset \mathbb{N}$ and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} 1_{\{j \in B\}}$$

exists, then this limit is called the *asymptotic frequency* of B. If π is a partition of $\mathbb{N}$, let $\Lambda(\pi)$ be the sequence consisting of the asymptotic frequencies of the blocks of π ranked in decreasing order (whenever these frequencies exist). Then $(\Lambda(\Pi(t)), t \geqslant 0) =_d (Y(t), t \geqslant 0)$.

The Aldous–Pitman fragmentation has arisen in a variety of contexts. Aldous and Pitman showed in [4] that if $X(t) = Y(\mathrm{e}^{-t})$, then the process $(X(t), -\infty < t < \infty)$ is a version of the *standard additive coalescent*. Loosely speaking, the standard additive coalescent is a coalescent process with the property that fragments of masses x and y are merging together at the rate $x + y$. See [20], [5], and [10] for more results related to the additive coalescent. Chassaing and Louchard [18] related the process $(Y(t), t \geqslant 0)$ to parking functions in combinatorics. Also, Bertoin [8, 10] showed that $(Y(t), t \geqslant 0)$ can be constructed from a Brownian motion with drift and that the so-called *eternal* versions of the additive coalescent could be constructed in a similar way from excursions of processes with exchangeable increments. Miermont [25] used this method to generalize [8] by studying a larger class of fragmentation processes, related to the additive coalescent, which can be obtained by adding drift to a general Lévy process with no positive jumps, implying several explicit laws for certain versions of the additive coalescent. The use of the ballot theorem therein was motivated by a similar approach of Schweinsberg [29] to analyze some functionals of the Brownian excursion.

The starting point for the present paper is the following theorem due to Aldous and Pitman, which shows three ways in which the Aldous–Pitman fragmentation is related to the stable subordinator of index $1/2$.

Theorem 1. *Let $(Y(t),\, t \geqslant 0)$ be the Aldous–Pitman fragmentation, and write the components of the fragmentation as $Y(t) = (Y_1(t), Y_2(t), \dots)$. Also, let $(\Pi(t),\, t \geqslant 0)$ be a partition-valued fragmentation with the property that $(\Lambda(\Pi(t)),\, t \geqslant 0) =_d (Y(t),\, t \geqslant 0)$. Let $Y^*(t)$ be the asymptotic frequency of the block of $\Pi(t)$ containing the integer 1. Let $(\sigma_{1/2}(t),\, t \geqslant 0)$ be a stable subordinator of index $1/2$. Then, the following hold:*

1. For every $t \geqslant 0$, we have $Y(t) =_d (J_1, J_2, \dots \mid \sigma_{1/2}(t) = 1)$, where $J_1, J_2, \dots$ are the jump sizes of $\sigma_{1/2}$ up to time t, ranked in decreasing order.

2. We have
$$\big(Y^*(t),\, t \geqslant 0\big) =_d \left(\frac{1}{1+\sigma_{1/2}(t)},\, t \geqslant 0\right).$$

3. As $t \to 0$, we have $t^{-2}(1 - Y_1(t), Y_2(t), Y_3(t), \dots) \to_d (\sigma_{1/2}(1), J_1, J_2, \dots)$.

Part 1 of the theorem can easily be obtained from Theorem 4 of [4] and scaling properties of stable subordinators. Part 2 is Theorem 6 of [4]. Part 3 is Corollary 13 of [4].

It is natural to ask whether there are other fragmentation processes related to the stable subordinator of index $\alpha \in (0,1)$ in the same ways that the Aldous–Pitman fragmentation is related to the stable subordinator of index $1/2$. In [11], Bertoin constructed a family of fragmentation processes, called self-similar fragmentations, which satisfy a scaling property. Because the Aldous–Pitman fragmentation is self-similar, one might expect the family of self-similar fragmentations to include fragmentations with properties that generalize properties of the Aldous–Pitman fragmentation. The purpose of this paper is to consider separately the three parts of Theorem 1 and to determine whether there are other self-similar fragmentations for which similar results hold, with the stable subordinator of index $1/2$ replaced by the stable subordinator of index α. Our conclusion, made precise by Theorem 2 and Propositions 1 and 2 below, is that only part 3 relating to asymptotics as $t \to 0$ can be easily generalized. Parts 1 and 2 of the theorem describe special properties of the $\alpha = 1/2$ case which do not extend, at least not in the most natural way, to other $\alpha \in (0,1)$.

Before stating these results, we will define self-similar fragmentations and review some of their properties. For $0 \leqslant l \leqslant 1$, define $\Delta_l = \{(x_i)_{i=1}^\infty : x_1 \geqslant x_2 \geqslant \cdots \geqslant 0, \sum_{i=1}^\infty x_i \leqslant l\}$. Note that $\Delta = \Delta_1$. We will denote points in Δ by $x = (x_1, x_2, \dots)$. Suppose $\kappa_t(l)$ is a probability measure on Δ_l for all $0 \leqslant l \leqslant 1$ and $t \geqslant 0$. For each $L = (l_1, l_2, \dots) \in \Delta$, let $\kappa_t(L)$ denote the distribution of the decreasing rearrangement of the terms of independent sequences $L_1, L_2, \dots$, where L_i has the distribution $\kappa_t(l_i)$ for all i. For each $t \geqslant 0$, denote by κ_t the family of distributions $(\kappa_t(L), L \in \Delta)$, which we call the *fragmentation kernel* generated by $(\kappa_t(l), 0 \leqslant l \leqslant 1)$. A time-homogeneous, Δ-valued Markov process whose transition semigroup is given by fragmentation kernels is called a *fragmentation process* or *ranked fragmentation*. This definition is essentially taken from [8], although we allow the sum of the masses of the fragments to decrease over time as in [6].

For $0 \leqslant l \leqslant 1$, let $g_l : \Delta \to \Delta_l$ be the map defined by $g_l(x_1, x_2, \dots) = (lx_1, lx_2, \dots)$. A ranked fragmentation is said to be a *homogeneous fragmentation* if, for all $0 \leqslant l \leqslant 1$ and $t > 0$, the probability measure $\kappa_t(l)$ is the image under g_l of the probability measure $\kappa_t(1)$. Notice that the term "homogeneous" does not refer to the assumed homogeneous Markov property of the semigroup. We call the fragmentation process a *self-similar fragmentation of index* $\beta \in \mathbb{R}$ if, for all $0 \leqslant l \leqslant 1$ and $t > 0$, $\kappa_t(l)$ is the image under g_l of $\kappa_r(1)$, where $r = tl^\beta$. Note that a self-similar fragmentation of index 0 is a homogeneous fragmentation.

Bertoin formulated definitions of homogeneity and self-similarity for partition-valued fragmentations that are analogous to the definitions given above for ranked fragmentations. In [9], Bertoin showed that all homogeneous partition-valued fragmentations can be described in terms of an erosion rate $c \geqslant 0$ and a measure ν on $\Delta \setminus (1, 0, 0, \dots)$, called the Lévy measure (or dislocation measure), which satisfies

$$\int_\Delta (1 - x_1)\, \nu(\mathrm{d}x) < \infty. \tag{1}$$

In [11], Bertoin showed that all self-similar fragmentations can be obtained from homogeneous fragmentations by a random time change which is determined by β. Consequently, all self-similar partition-valued fragmentation are fully described by their *characteristics* (β, c, ν). For each triple (β, c, ν), Bertoin constructs a self-similar fragmentation with these characteristics from a Poisson process. We will present this construction in the next section. The erosion rate c describes the rate at which singletons break away from larger blocks of the partition, and the Lévy measure governs the rates of other fragmentation events. If $\nu(\{x : x_1 + x_2 < 1\}) = 0$, then no block will break into more than two blocks at any given time. We then call the process a *binary* fragmentation.

If $(\Pi(t),\, t \geqslant 0)$ is a self-similar partition-valued fragmentation, then $\Pi(t)$ is an exchangeable random partition for all t. It follows from results of Kingman [23] that almost surely each block of $\Pi(t)$ has an asymptotic frequency. By Theorem 3 of [9], we have the stronger result that if $(\Pi(t),\, t \geqslant 0)$ is homogeneous, then almost surely all blocks of $\Pi(t)$ have asymptotic frequencies for all t. We can then see from the construction described in section 3 of [11] (and recalled in Sect. 2 below) that there exists a version $(\Pi(t),\, t \geqslant 0)$ of any self-similar fragmentation process such that almost surely all blocks of $\Pi(t)$ have asymptotic frequencies for all t. Furthermore, if $(\Pi(t),\, t \geqslant 0)$ denotes this version of a self-similar partition-valued fragmentation (which we will al-ways suppose in the sequel when considering self-similar fragmentations), then $(\Lambda(\Pi(t)),\, t \geqslant 0)$ is a self-similar ranked fragmentation with the same index of self-similarity. Berestycki [6] showed conversely that if $(X(t),\, t \geqslant 0)$ is a self-similar ranked fragmentation, then there exists a self-similar partition-valued fragmentation $(\Pi(t),\, t \geqslant 0)$ such that $(\Lambda(\Pi(t)),\, t \geqslant 0) = (X(t),\, t \geqslant 0)$. Consequently, self-similar ranked fragmentations are also in one-to-one correspondence with triples (β, c, ν), where $\beta \in \mathbb{R}$, $c \geqslant 0$, and ν is a measure on $\Delta \setminus (1, 0, 0, \dots)$ satisfying (1). Thus, we may work either with partition-valued fragmentations or ranked fragmentations, and both will be useful later in the paper.

Several examples of self-similar fragmentations have been studied. In [16] and [17], Brennan and Durrett studied a family of self-similar fragmentations. In the same context, see also Filippov [21]. Bertoin [11] considered an example that is related to Brownian excursions. Bertoin also observed in [11] that the Aldous–Pitman fragmentation is the binary self-similar fragmentation with characteristics $(1/2, 0, \nu)$, where the restriction of ν to the first coordinate has density $h(x) = (2\pi)^{-1/2} x^{-3/2} (1-x)^{-3/2} 1_{[1/2,1]}(x)$.

The following theorem, which is our main result, is related to part 1 of Theorem 1 about one-dimensional distributions. Here, and throughout the rest of the paper, $\sigma_\alpha = (\sigma_\alpha(t),\, t \geqslant 0)$ denotes a stable subordinator of index α.

Theorem 2. *Let $(X(t),\, t \geqslant 0)$ be a self-similar fragmentation, and let $\alpha \in (0,1)$. Let $J_1(t) \geqslant J_2(t) \geqslant \dots$ be the ranked jump sizes of σ_α between times 0 and t. If*

$$X(t) =_d \big(J_1(t), J_2(t), \cdots \,\big|\, \sigma_\alpha(t) = 1\big) \tag{2}$$

for all t, then $\alpha = 1/2$ and $(X(t),\, t \geqslant 0)$ is the Aldous–Pitman fragmentation.

The distributions on the right-hand side of (2) are part of a larger family of distributions studied in [28, 26]. Suppose $J_1 \geqslant J_2 \geqslant \dots$ is the ranked sequence of points from a Poisson process with intensity measure Θ on $(0, \infty)$, where Θ has density $\theta(x)$ and integrates $1 \wedge x$. Let $T = \sum_{i=1}^\infty J_i$. Then $(J_i/T)_{i=1}^\infty$ is a random point in Δ. Its distribution is called the *Poisson–Kingman distribution with Lévy density* θ and is denoted by $\mathrm{PK}(\theta)$. The conditional distribution of $(J_i/T)_{i=1}^\infty$ given $T = t$ is denoted by $\mathrm{PK}(\theta \,|\, t)$. Since $\sigma_\alpha(t) =_d t^{1/\alpha} \sigma_\alpha(1)$ by scaling properties of stable subordinators, we have

$$\big(J_1(t), J_2(t), \cdots \,\big|\, \sigma_\alpha(t) = 1\big) =_d \big(t^{1/\alpha} J_1(1), t^{1/\alpha} J_2(1), \cdots \,\big|\, \sigma_\alpha(1) = t^{-1/\alpha}\big). \tag{3}$$

For $\alpha \in (0,1)$, let θ_α be the Lévy density given by $\theta_\alpha(x) = C_\alpha\, x^{-\alpha-1}$, where C_α is the constant defined later in (8). If $J_1(t) \geqslant J_2(t) \geqslant \dots$ are the ranked jump sizes of σ_α between times 0 and t, then the distribution of $(J_1, J_2, \dots)$ has the same distribution as the ranked sequence of points of a Poisson point process with Lévy density $t\theta_\alpha$. Therefore (3) implies that the $\mathrm{PK}(t\theta_\alpha \,|\, 1)$ distribution is the same as the $\mathrm{PK}(\theta_\alpha \,|\, t^{-1/\alpha})$ distribution. Theorem 1 therefore shows that if $(Y(t),\, t \geqslant 0)$ is the Aldous–Pitman fragmentation, then $Y(t)$ has the

$\mathrm{PK}(\theta_{1/2}\,|\,t^{-2})$ distribution. Theorem 2 shows that there is no self-similar fragmentation $(X(t),\, t \geqslant 0)$ such that the distribution of $X(t)$ is $\mathrm{PK}(\theta_\alpha\,|\,t^{-1/\alpha})$ for all t. We have not, however, ruled out the possibility that a fragmentation which is not self-similar may have this property. In general, it remains an open problem to characterize the Lévy densities θ for which there exists a fragmentation process $(Z(t),\, t \geqslant 0)$ and a function $f : (0,\infty) \to (0,\infty)$ such that $Z(t)$ has the $\mathrm{PK}(\theta\,|\,f(t))$ distribution for all $t > 0$. However, we note that Miermont, in [25], has studied fragmentation processes that are not self-similar whose one-dimensional distributions are those of jump sizes for conditioned subordinators with varying Lévy measure, and one can show that a subclass of these fragmentations satisfy the asymptotics (4) below.

We now turn to a result for partition-valued fragmentations that pertains to the distribution of the mass of the block containing 1, which we sometimes call a "tagged fragment". The distribution of the mass of this block at time t is the same as the distribution of a size-biased pick from the sizes of the fragments of the corresponding ranked fragmentation at time t, provided that the sum of the sizes of the fragments at time t is 1 almost surely.

Proposition 1. *Let $(\Pi(t),\, t \geqslant 0)$ be a partition-valued binary self-similar fragmentation. Let $\alpha \in (0,1)$. Let $\lambda(t)$ be the asymptotic frequency of the block of $\Pi(t)$ containing the integer 1. If for some decreasing function g,*

$$\big(\lambda(t),\, t \geqslant 0\big) =_d \big(g\big(\sigma_\alpha(t)\big),\, t \geqslant 0\big),$$

then $\alpha = 1/2$, $g(x) = (1+Kx)^{-1}$ for some $K > 0$ and $(\Lambda(\Pi(t)),\, t \geqslant 0)$ is the Aldous–Pitman fragmentation, up to a multiplicative time constant.

Our next result gives, for each $\alpha \in (0,1)$, a family of binary self-similar fragmentations whose asymptotics as $t \to 0$ are related to the stable subordinator of index α.

Proposition 2. *Fix $\alpha \in (0,1)$, and let $C_\alpha = \alpha/(\Gamma(1-\alpha)\cos(\pi\alpha/2))$. Let ν be a Lévy measure on Δ such that $\nu(\{x : x_1 + x_2 < 1\}) = 0$ and the restriction ν_2 of ν to the second coordinate has density h, where*

$$h(x) = C_\alpha\, x^{-1-\alpha} s(x) 1_{[0,1/2]}(x)$$

for some positive function s satisfying $\lim_{x\to 0} s(x) = 1$. Let $\beta \geqslant 0$. Let $(X(t),\, t \geqslant 0)$ be the self-similar fragmentation with characteristics $(\beta, 0, \nu)$. Write $X(t) = (X_1(t), X_2(t), \dots)$. Then, as $t \to 0$, we have

$$t^{-1/\alpha}\big(1 - X_1(t), X_2(t), X_3(t), \dots\big) \to_d \big(\sigma_\alpha(1), J_1(1), J_2(1), \dots\big), \qquad (4)$$

where $J_1(1) \geqslant J_2(1) \geqslant \dots$ are the jump sizes of σ_α up to time 1.

Another connection between the self-similar fragmentations in Proposition 2 and stable subordinators can be deduced from Bertoin's work [13] regarding the small masses in self-similar fragmentations. Consider a binary self-similar fragmentation $(X(t),\, t \geqslant 0)$ with characteristics $(\beta, 0, \nu)$, where $\beta \geqslant 0$. Let ν_2 be the restriction of ν to the second coordinate. Let

$$N(\varepsilon, t) = \max\{i : X_i(t) > \varepsilon\}$$

be the number of components in the fragmentation at time t whose size is greater than ε. Let

$$M(\varepsilon, t) = \sum_{i=1}^{\infty} X_i(t) 1_{\{X_i(t) < \varepsilon\}}$$

be the total mass of the fragments at time t of size less than ε. Define $\phi(\varepsilon) = \nu_2([\varepsilon, 1/2])$ and $f(\varepsilon) = \int_0^\varepsilon x\, \nu_2(\mathrm{d}x)$. It follows from Theorem 1 of [13] that ϕ is regularly varying as $\varepsilon \downarrow 0$ with index $-\alpha$ if and only if f is regularly varying as $\varepsilon \downarrow 0$ with index $1-\alpha$. It also follows from Theorem 1 of [13] that if these regular variation conditions hold and $\beta = 1-\alpha$, then for all $t > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{N(\varepsilon, t)}{\phi(\varepsilon)} = \lim_{\varepsilon \downarrow 0} \frac{M(\varepsilon, t)}{f(\varepsilon)} = t$$

with probability one. Therefore, a straightforward calculation shows that if $(X(t),\, t \geqslant 0)$ satisfies the conditions of Proposition 2 with $\beta = 1-\alpha$, then $N(\varepsilon, t) \sim C_\alpha\, \alpha^{-1} t \varepsilon^{-\alpha}$ and $M(\varepsilon, t) \sim C_\alpha (1-\alpha)^{-1} t \varepsilon^{1-\alpha}$ with probability one for all $t > 0$, where $\sim$ means that the ratio of the two sides tends to 1 as $\varepsilon \downarrow 0$. For a stable subordinator of index α with Lévy measure $\eta(\mathrm{d}x) = C_\alpha\, x^{-1-\alpha}\, \mathrm{d}x$, the expected number of jumps of size larger than ε before time t is $C_\alpha\, \alpha^{-1} t \varepsilon^{-\alpha}$, and the expected value of the sum of the sizes of the jumps of size less than ε before time t is $C_\alpha (1-\alpha)^{-1} t \varepsilon^{1-\alpha}$. Thus, $N(\varepsilon, t)$ behaves like the number of jumps of a stable subordinator of index α that have size larger than ε, while $M(\varepsilon, t)$ behaves like the sum of the sizes of the small jumps of a stable subordinator of index α.

The rest of this paper is organized as follows. In Sect. 2, we present the Poisson process construction of self-similar fragmentations given by Bertoin in [11]. In Sect. 3, we establish some relevant facts about stable subordinators. In Sect. 4, we relate the small-time behavior of self-similar fragmentations to the dislocation measure (Proposition 3) and prove Proposition 2. We review some of Bertoin's results on the large-time behavior of self-similar fragmentations in Sect. 5. Section 6 is devoted to the proof of Theorem 2, and Sect. 7 is devoted to the proof of Proposition 1.

2 A Poisson process construction of self-similar fragmentations

In [11], Bertoin shows how to construct an arbitrary partition-valued self-similar fragmentation with characteristics (β, c, ν) from a Poisson process.

The conventions we are using here (for labelling partitions, and for taking reduced partitions in property 3 below) are actually those used in [6], but by exchangeability arguments explained therein they do indeed give the same distributional object as the construction in [9, 11].

Let ε_n be the partition of $\mathbb{N}$ into the two blocks $\{n\}$ and $\mathbb{N} \setminus \{n\}$. Given $x = (x_1, x_2, \dots) \in \Delta$, let P^x be the distribution of the random partition Π obtained by first defining an i.i.d. sequence of random variables $(Z_i)_{i=1}^{\infty}$ such that $P(Z_i = j) = x_j$ and $P(Z_i = 0) = 1 - \sum_{j=1}^{\infty} x_j$, and then defining Π to be the partition with the property that i and j are in the same block if and only if $Z_i = Z_j \geqslant 1$. Let κ be the measure on the set $\mathcal{P}$ of partitions of $\mathbb{N}$ defined such that for all Borel subsets B of $\mathcal{P}$, we have

$$\kappa(B) = \int_{\Delta} P^x(B)\, \nu(\mathrm{d}x) + c \sum_{i=1}^{\infty} 1_{\{\varepsilon_n \in B\}}. \tag{5}$$

Now, let $\#$ denote counting measure on $\mathbb{N}$, and let $((\Gamma_t, k_t),\, t \geqslant 0)$ be a Poisson point process on $\mathcal{P} \times \mathbb{N}$ with intensity measure $\kappa \otimes \#$. We can use this Poisson point process to construct a partition-valued self-similar fragmentation with characteristics (β, c, ν). The first step is to construct a homogeneous fragmentation with characteristics $(0, c, \nu)$. Let A_N consist of all partitions in $\mathcal{P}$ such that not all the integers $\{1, \dots, N\}$ are in the same block. Then $\kappa(A_N) < \infty$ for all N, so $(\Gamma_t, k_t) \in A_N \times \{1, \dots, N\}$ for only a discrete set of times, which we can enumerate as $t_1 < t_2 < \dots$. Define $(\Pi_N(t),\, t \geqslant 0)$ to be the unique process taking its values in the set of partitions of $\{1, \dots, N\}$ that satisfies the following three properties:

1. $\Pi_N(0)$ is the trivial partition of $\{1, \dots, N\}$.
2. Π_N is constant on $[t_{i-1}, t_i)$ for all $i \in \mathbb{N}$, where we set $t_0 = 0$.
3. Integers i and j are in distinct blocks of $\Pi_N(t_i)$ if and only if either i and j are in distinct blocks of $\Pi_N(t_{i-1})$, or i and j are in distinct blocks of Γ_{t_i} and both i and j are in a block of $\Pi_N(t_{i-1})$ whose smallest integer is k_{t_i}.

If π is a random partition of $\mathbb{N}$, let $R_N \pi$ be the random partition of $\{1, \dots, N\}$ such that if $1 \leqslant i, j \leqslant N$, then i and j are in the same block of $R_N \pi$ if and only if they are in the same block of π. The processes Π_N are consistent as N varies, so there exists a unique process $(\Pi(t),\, t \geqslant 0)$ such that $(R_N \Pi(t),\, t \geqslant 0) = (\Pi_N(t),\, t \geqslant 0)$ for all N. Then $(\Pi(t),\, t \geqslant 0)$ is a homogeneous fragmentation with characteristics $(0, c, \nu)$, as discussed in [9].

In [11], Bertoin shows that any self-similar fragmentation can be constructed from a homogeneous fragmentation by a random time change. Let $I_n(t)$ be the asymptotic frequency of the block of $\Pi(t)$ containing n. Define

$$T_n^{(\beta)}(t) = \inf\left\{ u \geqslant 0 : \int_0^u |I_n(r)|^{-\beta}\, \mathrm{d}r > t \right\}. \tag{6}$$

Define the process $(\Pi^{(\beta)}(t),\, t \geqslant 0)$ such that i and j are in the same block of $\Pi^{(\beta)}(t)$ if and only if i and j are in the same block of $\Pi(T_i^{(\beta)}(t))$. It is shown in [11] that $(\Pi^{(\beta)}(t),\, t \geqslant 0)$ is a self-similar partition-valued fragmentation with characteristics (β, c, ν). Therefore, $(\Lambda(\Pi^{(\beta)}(t)),\, t \geqslant 0)$ is a self-similar ranked fragmentation with characteristics (β, c, ν).

3 Stable subordinators

An $\mathbb{R}$-valued stochastic process $X = (X_t,\, t \geqslant 0)$ is called a *subordinator* if it is nondecreasing and has stationary independent increments. If X is a subordinator, then for all $\lambda \geqslant 0$, we have

$$E\big[\mathrm{e}^{-\lambda X_t}\big] = \exp\bigg(-t\bigg(d\lambda + \int_0^\infty \big(1-\mathrm{e}^{-\lambda x}\big)\, \eta(\mathrm{d}x)\bigg)\bigg),$$

where $d \geqslant 0$ is the *drift coefficient* and η is the *Lévy measure* on $(0,\infty)$, which must satisfy $\int_0^\infty (1 \wedge x)\, \eta(\mathrm{d}x) < \infty$. The process X is said to be a stable subordinator of index $\alpha \in (0,1)$ if $d = 0$ and

$$\eta(\mathrm{d}x) = C_\alpha\, x^{-1-\alpha}\, \mathrm{d}x \tag{7}$$

for some constant C_α. Since changing the constant C_α just changes time by a constant factor, we lose no generality by considering just one value for C_α. We will therefore take

$$C_\alpha = \frac{\alpha}{\Gamma(1-\alpha)\cos(\pi\alpha/2)}\,. \tag{8}$$

We will denote by $(\sigma_\alpha(t),\, t \geqslant 0)$ a subordinator whose Lévy measure is given by (7) and (8). The stable subordinator of index α satisfies the scaling property

$$\big(\lambda^{1/\alpha}\sigma_\alpha(t),\, t \geqslant 0\big) =_d \big(\sigma_\alpha(\lambda t),\, t \geqslant 0\big) \qquad \text{for every } \lambda > 0. \tag{9}$$

It is shown, for example, in chapter 17 of [22] that the characteristic function of $\sigma_\alpha(1)$ is given by

$$\phi(t) = \exp\Big(-|t|^\alpha\Big(1 - \mathrm{i}\ \mathrm{sgn}(t)\tan\Big(\frac{\pi\alpha}{2}\Big)\Big)\Big).$$

(Proposition 11 in chapter 17 of [22] actually gives this result when $C_\alpha = 2\alpha\Gamma(\alpha)\sin(\pi\alpha/2)/\pi$, but this is equivalent to (8) because of the duplication formula $\Gamma(\alpha)\Gamma(1-\alpha)\sin(\pi\alpha/2)\cos(\pi\alpha/2) = \pi/2$ for all $\alpha \in (0,1)$.) Let f_t be the density function of $\sigma_\alpha(t)$, and let $f = f_1$. It follows from the formulas given in [30] that if $A = \alpha^{1/2(1-\alpha)}(\cos(\pi\alpha/2))^{-1/(2(1-\alpha))}[2\pi(1-\alpha)]^{-1/2}$ and $B = (1-\alpha)\alpha^{\alpha/(1-\alpha)}(\cos(\pi\alpha/2))^{-1/(1-\alpha)}$, then

$$f(x) \sim Ax^{-1-\alpha/(2(1-\alpha))}\exp\big(-Bx^{-\alpha/(1-\alpha)}\big), \tag{10}$$

where $\sim$ means that the ratio of the two sides goes to 1 as $x \to 0$.

To get asymptotics for large x, note that [30] gives

$$f(x) = \sum_{n=1}^{\infty} a_n x^{-1-\alpha n},$$

where

$$a_n = \frac{(-1)^{n-1}\Gamma(n\alpha+1)}{n!\,\pi}\left(1+\tan^2\left(\frac{\pi\alpha}{2}\right)\right)^{n/2}\sin(n\pi\alpha). \tag{11}$$

Stirling's formula gives $\lim_{n\to\infty} a_n = 0$, so there exists a constant D such that if we write

$$f(x) = a_1 x^{-1-\alpha}\big(1 + r(x)\big), \tag{12}$$

then $|r(x)| \leqslant Dx^{-\alpha}$ for all x.

It is well-known that σ_α is a pure-jump process. The sequence consisting of the jump sizes of σ_α between times 0 and t, ranked in decreasing order, has the same distribution as the ranked sequence of points from a Poisson random measure on $(0,\infty)$ with intensity measure $\rho_t(x)\,\mathrm{d}x$, where $\rho_t(x) = C_\alpha\, t x^{-1-\alpha}$. It will be useful to consider size-biased picks from the jump sizes of σ_α. We will use the following lemma, which can be deduced from Lemma 2.1 of [27].

Lemma 1. *Fix $t > 0$. Let $J_1(t) \geqslant J_2(t) \geqslant \dots$ be the jump sizes of σ_α between times 0 and t. Let $J_1^*(t)$ be a size-biased pick from these jump sizes, and then let $J_2^*(t)$ be a size-biased pick from the remaining jump sizes. Then,*

$$P\big(J_1^* \in \mathrm{d}x \;\big|\; \sigma_\alpha(t) = z\big) = \frac{x\rho_t(x) f_t(z-x)}{z f_t(z)}\,\mathrm{d}x,$$

and the joint density of $(\sigma_\alpha(t), J_1^*(t), J_2^*(t))$ is given by

$$h(z,x,y) = \frac{\big(x\rho_t(x)\big)\big(y\rho_t(y)\big) f_t(z-x-y)}{z(z-x)}\,. \tag{13}$$

This Lemma implies the following result about the distribution as $t \to \infty$ of a size-biased pick from the jump sizes of $\sigma_\alpha(t)$, conditional on $\sigma_\alpha(t) = 1$.

Lemma 2. *Let $J_1^*(t)$ be a size-biased pick from the jump sizes of σ_α between times 0 and t. Let μ_t denote the conditional distribution of $t^{1/(1-\alpha)} J_1^*(t)$ given $\sigma_\alpha(t) = 1$. As $t \to \infty$, μ_t converges weakly to the Gamma$(1-\alpha, B\alpha/(1-\alpha))$ distribution.*

Proof. It follows from Lemma 1 that $P(J_1^*(t) \in \mathrm{d}x \,|\, \sigma_\alpha(t) = 1) = g_t(x)\,\mathrm{d}x$, where the density g_t is given by

$$g_t(x) = \frac{x\rho_t(x) f_t(1-x)}{f_t(1)} = \frac{C_\alpha\, t x^{-\alpha} f_t(1-x)}{f_t(1)} \tag{14}$$

for $x \in (0,1)$. It follows from (9) that $f_t(x) = f(t^{-1/\alpha}x)t^{-1/\alpha}$ for all $x > 0$. Using this fact and (14), we see that μ_t has density

$$\begin{aligned} h_t(x) &= g_t\big(t^{-1/(1-\alpha)}x\big)t^{-1/(1-\alpha)} \\ &= \frac{C_\alpha\, t\big(t^{-1/(1-\alpha)}x\big)^{-\alpha} f\big(t^{-1/\alpha}(1-t^{-1/(1-\alpha)}x)\big)t^{-1/(1-\alpha)}}{f\big(t^{-1/\alpha}\big)} \\ &= \frac{C_\alpha\, x^{-\alpha} f\big(t^{-1/\alpha}(1-t^{-1/(1-\alpha)}x)\big)}{f\big(t^{-1/\alpha}\big)} \end{aligned}$$

for $0 < x < t^{1/(1-\alpha)}$. Using (10), it follows that for each $x > 0$, we have

$$\begin{aligned} \lim_{t\to\infty} h_t(x) &= \lim_{t\to\infty} \frac{C_\alpha\, x^{-\alpha}\big(1-t^{-1/(1-\alpha)}x\big)^{-1-\alpha/(2(1-\alpha))}}{\exp\big(-Bt^{1/(1-\alpha)}\big)} \\ &\quad \times \exp\Big(-B\big(t^{-1/\alpha}(1-t^{-1/(1-\alpha)}x)\big)^{-\alpha/(1-\alpha)}\Big) \\ &= \lim_{t\to\infty} C_\alpha\, x^{-\alpha} \exp\Big(-Bt^{1/(1-\alpha)}\big((1-t^{-1/(1-\alpha)}x)^{-\alpha/(1-\alpha)}-1\big)\Big) \\ &= C_\alpha\, x^{-\alpha} \mathrm{e}^{-B\alpha x/(1-\alpha)}. \end{aligned}$$

Note that if $\lambda = B\alpha/(1-\alpha)$, then $\lambda^{1-\alpha} = \alpha/\cos(\pi\alpha/2)$, and thus $C_\alpha = \lambda^{1-\alpha}/\Gamma(1-\alpha)$. Thus, h_t converges pointwise to the Gamma$(1-\alpha,\lambda)$ density as $t\to\infty$. The result of the lemma then follows from Scheffé's Theorem. □

If Z has a Gamma$(1-\alpha, B\alpha/(1-\alpha))$ distribution, then for all $r \geqslant 0$,

$$E[Z^r] = \frac{\Gamma(r+1-\alpha)}{\Gamma(1-\alpha)}\left(\frac{B\alpha}{1-\alpha}\right)^{-r} = \frac{\Gamma(r+1-\alpha)}{\Gamma(1-\alpha)}\left(\frac{\cos(\pi\alpha/2)}{\alpha}\right)^{r/(1-\alpha)}. \quad (15)$$

We will need these moments in Sect. 6.

We now consider small-time asymptotics.

Lemma 3. *Let $J_1(t) \geqslant J_2(t) \geqslant \dots$ be the jump sizes of σ_α between times 0 and t. Let $J_1^*(t)$ be a size-biased pick from these jump sizes. If A is a Borel subset of $[0,1-a]$ for some $a>0$, then*

$$\lim_{t\to 0} t^{-1}P\big(J_1^*(t)\in A \,\big|\, \sigma_\alpha(t)=1\big) = \int_A C_\alpha\, x^{-\alpha}(1-x)^{-1-\alpha}\,\mathrm{d}x. \quad (16)$$

If B is a Borel subset of $[1/2,1-a]$, then

$$\lim_{t\to 0} t^{-1}P\big(J_1(t)\in B \,\big|\, \sigma_\alpha(t)=1\big) = \int_B C_\alpha\, x^{-1-\alpha}(1-x)^{-1-\alpha}\,\mathrm{d}x. \quad (17)$$

Proof. For all $t>0$ and all Borel subsets A of $[0,1-a]$, Lemma 1 implies that

$$\begin{aligned} t^{-1}P\big(J_1^*(t)\in A \,\big|\, \sigma_\alpha(t)=1\big) &= \int_A \frac{x\rho_t(x)f_t(1-x)}{tf_t(1)}\,\mathrm{d}x \\ &= \int_A \frac{C_\alpha\, x^{-\alpha}f_t(1-x)}{f_t(1)}\,\mathrm{d}x. \end{aligned}$$

By (9), $f_t(1)t^{1/\alpha} = f(t^{-1/\alpha})$ and $f_t(1-x)t^{1/\alpha} = f(t^{-1/\alpha}(1-x))$. Therefore, (12) implies that $a_1 t^{1+1/\alpha}(1-Dt) \leqslant f_t(1)t^{1/\alpha} \leqslant a_1 t^{1+1/\alpha}(1+Dt)$ and $a_1 t^{1+1/\alpha}(1-x)^{-1-\alpha}(1-Dt(1-x)^{-\alpha}) \leqslant f_t(1-x)t^{1/\alpha} \leqslant a_1 t^{1+1/\alpha}(1-x)^{-1-\alpha}(1-Dt(1-x)^{-\alpha})$. It follows that for all $x \in [0,1-a]$, we have

$$(1-x)^{-1-\alpha}\left(\frac{1-Dt(1-a)^{-\alpha}}{1+Dt}\right) \leqslant \frac{f_t(1-x)}{f_t(1)} \leqslant (1-x)^{-1-\alpha}\left(\frac{1+Dt(1-a)^{-\alpha}}{1-Dt}\right).$$

Therefore, by the Dominated Convergence Theorem,

$$\lim_{t\to 0} t^{-1} P\big(J_1^*(t) \in A \mid \sigma_\alpha(t)=1\big) = \int_A C_\alpha\, x^{-\alpha}(1-x)^{-1-\alpha}\,\mathrm{d}x,$$

which is (16).

If $J_1^*(t) > 1/2$, then $J_1^*(t) = J_1(t)$. Therefore, it follows from the definition of a size-biased pick from a sequence that for $x \in [1/2, 1-a]$,

$$P\big(J_1(t) \in \mathrm{d}x \mid \sigma_\alpha(t)=1\big) = x^{-1} P\big(J_1^*(t) \in \mathrm{d}x \mid \sigma_\alpha(t)=1\big).$$

Therefore, if B is a Borel subset of $[1/2,1]$, then

$$t^{-1}P\big(J_1(t) \in B \mid \sigma_\alpha(t)=1\big) = \int_B \frac{\rho_t(x) f_t(1-x)}{t f_t(1)}\,\mathrm{d}x = \int_B \frac{C_\alpha\, x^{-1-\alpha} f_t(1-x)}{f_t(1)}\,\mathrm{d}x.$$

Equation (17) follows from the Dominated Convergence Theorem as in the proof of (16). $\square$

Lemma 4. *Let $J_1(t) \geqslant J_2(t) \geqslant \ldots$ be the jump sizes of σ_α between times 0 and t. Let $J_1^*(t)$ be a size-biased pick from these jump sizes, and then let $J_2^*(t)$ be a size-biased pick from the remaining jump sizes. Let A be a Borel subset of $[0,1]^2$ such that $A \subset \{(x,y) \in [0,1]^2 : 0 < x+y < 1-a\}$ for some $a > 0$. Then*

$$\lim_{t\to 0} t^{-2} P\big((J_1^*(t), J_2^*(t)) \in A \mid \sigma_\alpha(t)=1\big) = \int_A \frac{C_\alpha^2\, x^{-\alpha} y^{-\alpha}(1-x-y)^{-1-\alpha}}{1-x}\,\mathrm{d}x\,\mathrm{d}y. \quad (18)$$

Proof. Using (13), we see that

$$t^{-2}P\big((J_1^*(t), J_2^*(t)) \in A \mid \sigma_\alpha(t)=1\big) = \int_A \frac{t^{-2} x\rho_t(x) y\rho_t(y) f_t(1-x-y)}{(1-x)f_t(1)}\,\mathrm{d}x\,\mathrm{d}y = \int_A \frac{C_\alpha^2\, x^{-\alpha} y^{-\alpha} f_t(1-x-y)}{(1-x) f_t(1)}\,\mathrm{d}x\,\mathrm{d}y.$$

Equation (12) gives

$$(1-x-y)^{-1-\alpha}\left(\frac{1-Dt(1-a)^{-\alpha}}{1+Dt}\right) \leqslant \frac{f_t(1-x-y)}{f_t(1)} \leqslant (1-x-y)^{-1-\alpha}\left(\frac{1+Dt(1-a)^{-\alpha}}{1-Dt}\right).$$

The lemma now follows from the Dominated Convergence Theorem. □

4 Small-time behavior of self-similar fragmentations

The proofs of Theorem 2 and Proposition 1 will use results on the small-time behavior of self-similar fragmentations. In this section, we record some results that we will need, and then we prove Proposition 2. First we give a way to recover the dislocation measure ν of a self-similar fragmentation with positive index and no erosion from its semigroup.

Proposition 3. *Let $(X(t), t \geqslant 0)$ be a Δ-valued self-similar fragmentation with characteristics $(\beta, 0, \nu)$, where $\beta \geqslant 0$. For all $t > 0$, let μ_t be the measure on Δ defined by $\mu_t(A) = t^{-1}P(X(t) \in A)$ for all Borel measurable subsets A of Δ. Then μ_t converges weakly to ν as $t \to 0$ on any subset of Δ that is the complement of an open neighborhood of $(1, 0, 0, \dots)$.*

We will need the following lemma in the course of the proof:

Lemma 5. *Let $(\xi_t, t \geqslant 0)$ be a subordinator with Lévy mesure $L(\mathrm{d}x)$. Then the measure $t^{-1}P(\xi_t \in \mathrm{d}x)$ converges to $L(\mathrm{d}x)$ as $t \to 0$ weakly on any set of the form $(a, +\infty)$ with $a > 0$. Moreover, denoting the jump $\xi_u - \xi_{u-}$ at time u by $\delta\xi_u$, one has, as $t \to 0$,*

$$P\big(\xi_t \geqslant a \text{ and } \delta\xi_u < a \text{ for all } u \in [0,t]\big) = o(t).$$

Proof. The first part is classical, see e.g. [7]. For the second part, standard properties of Poisson measures give

$$P\big(\xi_t \geqslant a \text{ and } \delta\xi_u \geqslant a \text{ for some } u \in [0,t]\big) = tL\big([a,\infty)\big) + o(t). \tag{19}$$

On the other hand, the Portmanteau theorem (see [15]) and the first part imply

$$\limsup_{t\to 0} \frac{1}{t} P(\xi_t \geqslant a) \leqslant L\big([a,\infty)\big). \tag{20}$$

Hence, dividing (19) by t and subtracting from (20) gives

$$\limsup_{t\to 0} \frac{1}{t} P\big(\xi_t \geqslant a \text{ and } \delta\xi_u < a \text{ for all } u \in [0,t]\big) \leqslant 0.$$ □

Proof of Proposition 3. Let $A_\delta = \{x \in \Delta : x_1 \leqslant 1-\delta\}$. Any subset of Δ that is the complement of an open neighborhood of $(1,0,0,\dots)$ is a subset of A_δ for some $\delta > 0$. Therefore, it suffices to show that μ_t converges weakly to ν on A_δ for all $\delta > 0$. Fix $\delta > 0$, and let G be a positive, bounded, continuous function on Δ such that $G(x) = 0$ for $x \notin A_\delta$. By the definition of μ_t and the definition of weak convergence, we need to show that

$$\lim_{t \to 0} t^{-1} E\big[G\big(X(t)\big)\big] = \int_\Delta G(s)\,\nu(\mathrm{d}s). \tag{21}$$

Without loss of generality, suppose that $X(t) = (X_1(t), X_2(t), \dots) = \Lambda(\Pi^{(\beta)}(t))$ for a partition-valued fragmentation process $\Pi^{(\beta)}$ with the same characteristics as X. We may also assume that $\Pi^{(\beta)}$ is constructed by time-changing a partition-valued fragmentation Π with characteristics $(0,0,\nu)$ as in Sect. 2. That is, if $I_n(t)$ is the asymptotic frequency of the block of $\Pi(t)$ containing n and $T_n^{(\beta)}(t)$ is defined as in (6), then i and j are in the same block of $\Pi^{(\beta)}(t)$ if and only if i and j are in the same block of $\Pi(T_i^{(\beta)}(t))$. Also, we suppose that Π is constructed out of a Poisson point process $((\Gamma_t, k_t),\, t \geqslant 0)$ with intensity $\kappa \otimes \#$ as in Sect. 2. Notice that for every i and $t \geqslant 0$, we have $T_i^{(\beta)}(t) \leqslant t$ because $\beta > 0$. It follows that $(\Pi^{(\beta)}(u), 0 \leqslant u \leqslant t)$ is completely determined by the process $((\Gamma_u, k_u), 0 \leqslant u \leqslant t)$.

Let $(\Theta_t,\, t \geqslant 0)$ be the process such that $\Theta_t = \Gamma_t$ whenever (Γ_t, k_t) is a point of the Poisson process such that k_t is the least element of the block with maximal asymptotic frequency of $\Pi(t)$ at time $t-$. If two or more blocks are tied for having the largest asymptotic frequency, we rank the blocks according to their smallest elements. As a consequence of Lemma 10 in [6], Θ is a Poisson point process with intensity κ.

Let N_t be the cardinality of $\{s \in [0,t] : \Lambda(\Theta_s) \in A_\delta\}$. Note that N_t has a Poisson distribution with mean $t\nu(A_\delta)$. Therefore,

$$\lim_{t \to 0} t^{-1} E\big[G\big(X(t)\big)1_{\{N_t \geqslant 2\}}\big] \leqslant \lim_{t \to 0} t^{-1} \|G\|_\infty P(N_t \geqslant 2) = 0.$$

Next, note that $E[G(X(t))1_{\{N_t=0\}}] \leqslant \|G\|_\infty P(\{X_1(t) \leqslant 1-\delta\} \cap \{N_t = 0\})$. If π is a partition of $\mathbb{N}$, let $\Lambda_j(\pi)$ denote the asymptotic frequency of the block of π having the jth-largest asymptotic frequency. Since $\beta \geqslant 0$, we have

$$X_1(t) \geqslant \prod_{0 \leqslant u \leqslant t} \Lambda_1(\Theta_u) \geqslant 1 - \sum_{0 \leqslant u \leqslant t} \big(1 - \Lambda_1(\Theta_u)\big).$$

Since $t \mapsto \sum_{0 \leqslant u \leqslant t}(1 - \Lambda_1(\Theta_u))$ is a subordinator, it follows from Lemma 5 that $P(\{X_1(t) \leqslant 1-\delta\} \cap \{N_t = 0\}) = o(t)$. Therefore,

$$\lim_{t \to 0} t^{-1} E\big[G\big(X(t)\big)1_{\{N_t=0\}}\big] = 0.$$

Thus, to prove (21), it remains only to show that

$$\lim_{t\to 0} t^{-1} E\big[G\big(X(t)\big)1_{\{N_t=1\}}\big] = \int_\Delta G(s)\,\nu(\mathrm{d}s). \tag{22}$$

Let $0 < \varepsilon < 1/2$, and let $\eta > 0$. Then there exists a positive number t_0 such that $P(I_i(t_0) < 1-\varepsilon) < \eta$ for every $i \geqslant 1$. Fix $t < t_0$. On the event $\{N_t = 1\}$, define U such that $\Lambda(\Theta_{tU}) \in A_\delta$. Note that U has a uniform distribution on $[0,1]$. Define B to be the event that $U \leqslant (1-\varepsilon)^\beta$. Let B_0 be the event that $I_1(tU-) \geqslant 1-\varepsilon$. Fix $J \in \mathbb{N}$. For $1 \leqslant j \leqslant J$, let i_j be the smallest integer in the block of $\Pi(tU)$ having the jth-largest asymptotic frequency, provided that integer is in the same block as 1 at time $tU-$; otherwise, define $i_j = 0$. Let B_j be the event that either $i_j = 0$ or $|I_{i_j}(T^{(\beta)}_{i_j}(t)) - I_{i_j}(tU)| \leqslant \varepsilon$.

We have $P(B \mid N_t = 1) = (1-\varepsilon)^\beta$. Also,

$$P(B_0 \mid N_t = 1) \geqslant P\big(I_1(t) \geqslant 1-\varepsilon\big) \geqslant 1-\eta\,.$$

If B and B_0 occur, then

$$\int_0^{tU} I_1(s)^{-\beta}\,\mathrm{d}s \leqslant tU(1-\varepsilon)^{-\beta} \leqslant t,$$

which implies that $T^{(\beta)}_1(t) \geqslant tU$. If, in addition, $i_j > 0$, then $tU \leqslant T^{(\beta)}_{i_j}(t) \leqslant t$. In this case $|I_{i_j}(T^{(\beta)}_{i_j}(t)) - I_{i_j}(tU)| \leqslant |I_{i_j}(t) - I_{i_j}(tU)|$ which, conditional on B, B_0, and $N_t = 1$, is less than or equal to ε with probability at least $1-\eta$. Thus,

$$P(B \cap B_0 \cap B_1 \cap \cdots \cap B_J \mid N_t = 1) \geqslant (1-\varepsilon)^\beta - (J+1)\eta\,.$$

Suppose $B, B_0, B_1, \ldots, B_J$ all occur. If $i_j = 0$, then $X_j(t) < \varepsilon$ and $\Lambda_j(\Theta_{tU}) \leqslant \varepsilon/(1-\varepsilon)$, so $|X_j(t) - \Lambda_j(\Theta_{tU})| \leqslant 2\varepsilon$. If $i_j > 0$, then

$$\begin{aligned}\left|I_{i_j}\left(T^{(\beta)}_{i_j}(t)\right) - \Lambda_j(\Theta_{tU})\right| &\leqslant \left|I_{i_j}\left(T^{(\beta)}_{i_j}(t)\right) - I_{i_j}(tU)\right| + \left|I_{i_j}(tU) - \Lambda_j(\Theta_{tU})\right| \\ &\leqslant \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Since the block of $\Pi^{(\beta)}(t)$ containing the integer i_j has asymptotic frequency $I_{i_j}(T^{(\beta)}_{i_j}(t))$, it follows that $|X_j(t) - \Lambda_j(\Theta_{tU})| \leqslant 2\varepsilon$. Thus, for $t < t_0$,

$$P\big(|X_j(t) - \Lambda_j(\Theta_{tU})| \leqslant 2\varepsilon \text{ for } j = 1, \ldots, J \;\big|\; N_t = 1\big) \geqslant (1-\varepsilon)^\beta - (J+1)\eta\,.$$

By letting $\varepsilon, \eta \to 0$ and applying Theorem 3.1 of [15], we can see that the conditional distribution of $(X_1(t), \ldots, X_J(t))$ given $N_t = 1$ converges to the distribution of $(\Lambda_1(\Theta_{tU}), \ldots, \Lambda_J(\Theta_{tU}))$. By properties of weak convergence in Δ (see chapter 4 of [15]), it follows that the conditional distribution of $X(t)$ given $N_t = 1$ converges as $t \to 0$ to the distribution of $\Lambda(\Theta_{tU})$, which does not depend on t. Thus,

$$\begin{aligned}\lim_{t\to 0} t^{-1}E\big[G\big(X(t)\big)1_{\{N_t=1\}}\big] &= \lim_{t\to 0} t^{-1}P(N_t=1)\,E\big[G\big(X(t)\big)\bigm| N_t=1\big] \\ &= \lim_{t\to 0}\nu(A_\delta)\,\mathrm{e}^{-t\nu(A_\delta)}\,E\big[G\big(\Lambda(\Theta_{tU})\big)\big] \\ &= \lim_{t\to 0}\nu(A_\delta)\,\mathrm{e}^{-t\nu(A_\delta)}\int_\Delta \frac{G(s)}{\nu(A_\delta)}\,\nu(\mathrm{d}s) \\ &= \int_\Delta G(s)\,\nu(\mathrm{d}s),\end{aligned}$$

which is (22). □

Remark 1. In this proposition and the following corollary, the assumption that $c=0$, $\beta\geqslant 0$ could be avoided. When $c>0$, we may follow essentially the same reasoning as above because the drift at rate c has little effect on the block sizes for small t. When $\beta<0$, however, the proof requires a more careful analysis of the time-changes $T_i^{(\beta)}$. We thus omit the proof here, as we are only concerned with positive self-similarity indices.

From Proposition 3, we get the following result concerning the small-time behavior of the asymptotic frequency of the block containing 1 in a partition-valued fragmentation.

Corollary 1. *Let $(\Pi(t),\, t\geqslant 0)$ be a partition-valued self-similar fragmentation with characteristics (β,c,ν). Let $\lambda(t)$ be the asymptotic frequency of the block containing 1 at time t. For all $t>0$, let γ_t be the measure on $[0,1]$ defined by $\gamma_t(A)=t^{-1}P(\lambda(t)\in A)$ Let ν_i be the restriction of ν to the ith coordinate. Let γ be the measure on $[0,1]$ defined by*

$$\gamma(A)=\sum_{i=1}^{\infty}\int_A x\,\nu_i(\mathrm{d}x) \tag{23}$$

for all A. Then, γ_t converges weakly to γ as $t\to 0$ on $[a,1-a]$ for all $a>0$.

Proof. Let μ_t be the measure on Δ defined by $\mu_t(A)=t^{-1}P(\Lambda(\Pi(t))\in A)$ for all Borel measurable subsets A of Δ. Let $\mu_{t,i}$ be the restriction of μ_t to the ith coordinate. Then,

$$\gamma_t(A)=\sum_{i=1}^{\infty}\int_A x\,\mu_{t,i}(\mathrm{d}x).$$

Let f be a bounded continuous function defined on $[a,1-a]$. By Proposition 3, $\mu_{t,i}$ converges weakly on $[a,1-a]$ to ν_i for all i. Therefore,

$$\begin{aligned}\lim_{t\to 0}\int_a^{1-a} f(x)\,\gamma_t(\mathrm{d}x) &= \lim_{t\to 0}\sum_{i=1}^{\infty}\int_a^{1-a} xf(x)\,\mu_{t,i}(\mathrm{d}x) \\ &= \sum_{i=1}^{\infty}\lim_{t\to 0}\int_a^{1-a} xf(x)\,\mu_{t,i}(\mathrm{d}x) = \sum_{i=1}^{\infty}\int_a^{1-a} xf(x)\,\nu_i(\mathrm{d}x) \\ &= \int_a^{1-a} f(x)\,\gamma(\mathrm{d}x),\end{aligned}$$

which implies the conclusion of the corollary. Note that interchanging the limit and the sum is justified because $\mu_{t,i}([a, 1-a]) = 0$ for all t whenever $i > 1/a$, so only finitely many terms in the sum are nonzero. □

We now prove Proposition 2, which shows that the small-time behavior of some self-similar fragmentations is related to the stable subordinator of index α. In the case of homogeneous fragmentations, our results are similar to the results in section 4 of [6]. Our arguments are also similar to those in section 4 of [6], but we work here with partition-valued fragmentations rather than ranked fragmentations and prove the result for self-similar fragmentations with a positive index of self-similarity in addition to homogeneous fragmentations.

Proof of Proposition 2. Since the fragmentation $(X(t), t \geqslant 0)$ is a binary fragmentation with no erosion and positive index β, we have that $1 - X_1(t) = \sum_{i=2}^{\infty} X_i(t)$ for all t. Also, since σ_α is a pure-jump process, $\sigma_\alpha(1) = \sum_{i=1}^{\infty} J_i(1)$. Therefore, to show (4), it suffices to show that

$$t^{-1/\alpha}\big(X_2(t), X_3(t), \dots\big) \to_d \big(J_1(1), J_2(1), \dots\big).$$

Therefore, it suffices to show that

$$t^{-1/\alpha}\big(X_2(t), \dots, X_{n+1}(t)\big) \to_d \big(J_1(1), \dots, J_n(1)\big) \tag{24}$$

for all $n \in \mathbb{N}$.

As in the proof of Proposition 3, we may suppose that $X(t) = \Lambda(\Pi^{(\beta)}(t))$ for all t, where $\Pi^{(\beta)}$ is the partition-valued fragmentation with characteristics $(\beta, 0, \nu)$ that is obtained from a homogeneous framgmentation Π, being constructed out of a Poisson process $((\Gamma_t, k_t), t \geqslant 0)$ with intensity $\kappa \otimes \#$ as in Sect. 2.

For all $k \in \mathbb{N}$, let $(r_t^{(k)}, t \geqslant 0)$ be the Poisson point process on $[0, 1/2]$ with the property that $r_t^{(k)} = r$ if and only if $(\Gamma_t, k_t) = (\pi, k)$ for some $\pi \in \mathcal{P}$ such that the block of π with the second-largest asymptotic frequency has asymptotic frequency r. Note that for all k, the Poisson point process $(r_t^{(k)}, t \geqslant 0)$ has characteristic measure $\nu_2(\mathrm{d}x)$, where ν_2 is the restriction of ν to the second coordinate. For all j, let $K_j(t)$ be the jth-largest point of $(r_s^{(1)}, 0 \leqslant s \leqslant t)$. Let $\tau_j(t)$ be the time such that $r_{\tau_j(t)}^{(1)} = K_j(t)$. Let $N_j(t)$ be the smallest integer which is in the same block as 1 in the partition $\Pi(\tau_j(t)-)$ but is not in the same block as 1 in $\Pi(\tau_j(t))$.

Define another Poisson point process $(\Theta_t, t \geqslant 0)$ whose characteristic measure has density

$$q(x) = C_\alpha(s(x) \vee 1)x^{-1-\alpha}.$$

We now construct two new Poisson point processes by marking, as described in chapter 5 of [24]. Let $(\Theta_t^{(1)}, t \geqslant 0)$ consist of the marked points of $(\Theta_t, t \geqslant 0)$ when a point at x is marked with probability $1/(s(x) \vee 1)$. Let $(\Theta_t^{(2)}, t \geqslant 0)$

consist of the marked points of $(\Theta_t, t \geqslant 0)$ when a point at x is marked with probability $s(x)1_{[0,1/2]}(x)/(s(x) \vee 1)$. Then, $(\Theta_t^{(1)}, t \geqslant 0)$ is a Poisson point process whose characteristic measure has density $q_1(x) = C_\alpha\, x^{-1-\alpha}$, and $(\Theta_t^{(2)}, t \geqslant 0)$ is a Poisson point process whose characteristic measure has density $q_2(x) = C_\alpha\, x^{-1-\alpha} s(x) 1_{[0,1/2]}(x)$.

Let $L_j(t)$ denote the jth largest point of $(\Theta_s^{(1)}, 0 \leqslant s \leqslant t)$, and let $\widetilde{K}_j(t)$ denote the jth largest point of $(\Theta_s^{(2)}, 0 \leqslant s \leqslant t)$. If the n largest points of $(\Theta_s, 0 \leqslant s \leqslant t)$ are also points in both $(\Theta_s^{(1)}, 0 \leqslant s \leqslant t)$ and $(\Theta_s^{(2)}, 0 \leqslant s \leqslant t)$, then $L_j(t) = \widetilde{K}_j(t)$ for $j = 1, \ldots, n$. For all $x > 0$, the probability that the largest point of $(\Theta_s, 0 \leqslant s \leqslant t)$ is less than x approaches 1 as $t \to 0$. Since $\lim_{x \to 0} s(x) = 1$, we have $\lim_{x \to 0} s(x)1_{[0,1/2]}(x)/(s(x) \vee 1) = 1$ and $\lim_{x \to 0} 1/(s(x) \vee 1) = 1$. It follows from these observations that

$$\lim_{t \to 0} P\big(L_j(t) = \widetilde{K}_j(t) \text{ for } j = 1, \ldots, n\big) = 1. \tag{25}$$

Note that $(L_1(t), \ldots, L_n(t))$ has the same distribution as the sizes of the n largest jumps of $(\sigma_\alpha(s), 0 \leqslant s \leqslant t)$. By scaling properties of the stable subordinator of index α, it follows that

$$t^{-1/\alpha}\big(L_1(t), \ldots, L_n(t)\big) =_d \big(J_1(1), \ldots, J_n(1)\big). \tag{26}$$

Since $(\widetilde{K}_1(t), \ldots, \widetilde{K}_n(t)) =_d (K_1(t), \ldots, K_n(t))$, It follows from equations (25) and (26), and Theorem 3.1 in [15] that

$$t^{-1/\alpha}\big(K_1(t), \ldots, K_n(t)\big) \to_d \big(J_1(1), \ldots, J_n(1)\big) \tag{27}$$

as $t \to 0$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. We will show next that for all $n \in \mathbb{N}$, we have

$$\lim_{t \to 0} P\big(\big|t^{-1/\alpha} K_j(t) - t^{-1/\alpha} X_{j+1}(t)\big| < \varepsilon \text{ for } j = 1, \ldots, n\big) = 1. \tag{28}$$

Equations (27) and (28), combined with Theorem 3.1 of [15], establish (24), which suffices to prove Proposition 2.

Given $0 < \delta < 1/2$ and $i \in \mathbb{N}$, let λ_t^i be the asymptotic frequency of the set of all integers m such that m is in the same block as i in every partition π for which $\Gamma_s = \pi$ and $k_s = i$ for some $s \in [0, t]$. Let $A_{\delta,t}^i$ be the event that $\lambda_t^i > 1 - \delta$. If $A_{\delta,t}^1$ occurs, then the block of $\Pi(t)$ containing 1 has asymptotic frequency at least $1 - \delta$. Also, since $\beta \geqslant 0$, it follows that $T_i^\beta(t) \leqslant t$ for every $i \geqslant 1$. Therefore, taking $i = 1$, if $A_{\delta,t}^1$ occurs, the block of $\Pi^{(\beta)}(t)$ containing 1 has asymptotic frequency at least $1 - \delta$.

Let $B_{j,t}$ be the event that $\tau_j(t) \leqslant T_1^\beta(t)$. Note that this is the same as the event that $\tau_j(t) \leqslant T_{N_j(t)}^\beta(t)$ because 1 and $N_j(t)$ are in the same block before time $\tau_j(t)$. Suppose $A_{\delta,t}^1$ occurs, and suppose $A_{\delta,t}^{N_j(t)}$ and $B_{j,t}$ occur for $j = 1, \ldots, n$. Then, for $j = 1, \ldots, n$, the block of $\Pi(\tau_j(t)-)$ containing $N_j(t)$

has asymptotic frequency between $1-\delta$ and 1, the block of $\Pi(\tau_j(t))$ containing $N_j(t)$ has asymptotic frequency between $K_j(t)(1-\delta)$ and $K_j(t)$, and the asymptotic frequency of the block of $\Pi^{(\beta)}(t)$ containing $N_j(t)$ has asymptotic frequency between $K_j(t)(1-\delta)^2$ and $K_j(t)$. Furthermore, the largest of all blocks of $\Pi^{(\beta)}(t)$ not containing any of the integers $\{1, N_1(t), \ldots, N_n(t)\}$ has asymptotic frequency at most $\max\{\delta K_1(t), K_{n+1}(t)\}$. Indeed, this block could be obtained from the $n+1$-th largest fragmentation of the fragment containing 1, which since $\delta < 1/2$ is also the largest one, in which case its asymptotic frequency is at most $K_{n+1}(t)$. Alternatively, it could be obtained from one of the fragments containing some $N_j(t)$ for $1 \leqslant j \leqslant n$. Since we assume that $A_{\delta,t}^{N_j(t)}$ occurs for every $1 \leqslant j \leqslant n$, the size of these fragments can not be reduced by more than a factor of $1-\delta$. Therefore, at time t, the fragments that do not contain any of the $N_j(t)$, for $1 \leqslant j \leqslant n$, but are obtained by splitting the blocks containing one of the $N_j(t)$, $1 \leqslant j \leqslant n$, have asymptotic frequency smaller than $\delta K_j(t) \leqslant \delta K_1(t)$.

Therefore, if in addition $\delta K_1(t) < K_n(t)$, then

$$K_j(t)(1-\delta)^2 \leqslant X_{j+1}(t) \leqslant K_j(t) \tag{29}$$

for $j = 1, \ldots, n$.

Note that $\lim_{t\to 0} P(A_{\delta,t}^1) = 1$ for all $\delta \in (0, 1/2)$. Likewise, for all $j \in \mathbb{N}$ and $\delta \in (0, 1/2)$, we have $\lim_{t\to 0} P(A_{\delta,t}^{N_j(t)}) = 1$. We now prove that

$$\lim_{t\to 0} P(B_{j,t}) = 1. \tag{30}$$

Let $\varepsilon > 0$. Choose δ small enough that $1-(1-\delta)^\beta < \varepsilon/2$. Then choose t small enough that $P(A_{\delta,t}^1) > 1-\varepsilon/2$. Suppose $A_{\delta,t}^1$ occurs. Then the fragment of $\Pi(s)$ containing 1 has asymptotic frequency larger than $1-\delta$ for $0 \leqslant s \leqslant t$. It follows from (6) that

$$(1-\delta)^\beta t \leqslant T_1^\beta(t) \leqslant t.$$

Since $\tau_j(t)$ is uniform on $(0,t)$, we have $P(B_{j,t} \mid A_{\delta,t}^1) > 1-\varepsilon/2$. Since $P(A_{\delta,t}^1) > 1-\varepsilon/2$, it follows that $P(B_{j,t}) > 1-\varepsilon$, which implies (30). Last, by (27),

$$\lim_{\delta\to 0} \lim_{t\to 0} P\big(\delta K_1(t) < K_n(t)\big) = 1.$$

These results, combined with (29), prove (28). □

5 Large-time behavior of self-similar fragmentations

In [12], Bertoin studied the asymptotic behavior of self-similar fragmentations as $t \to \infty$. Using facts from [14] about semi-stable processes, he proved the following result.

Lemma 6. *Let* $(X(t), t \geqslant 0) = ((X_1(t), X_2(t), \dots), t \geqslant 0)$ *be a self-similar fragmentation with characteristics* (β, c, ν)*. Suppose* $\nu(\{x : \sum_{i=1}^{\infty} x_i < 1\}) = 0$*. Also assume that there exists no* $r > 0$ *such that the size of every fragment at time* $t > 0$ *lies in the set* $\{\mathrm{e}^{-kr} : k = 0, 1, \dots\}$*. For* $q \geqslant 0$*, define*

$$\Phi(q) = c(q+1) + \int_{\Delta} \left(1 - \sum_{i=1}^{\infty} x_i^{q+1}\right) \nu(\mathrm{d}x). \tag{31}$$

Assume that

$$\Phi'(0+) = c + \sum_{i=1}^{\infty} \int_{\Delta} x_i \log(1/x_i)\, \nu(\mathrm{d}x) < \infty. \tag{32}$$

If $\beta = 0$*, then* $\lim_{t\to\infty} t^{-1} \log(X_1(t))$ *exists and is finite almost surely. If* $\beta > 0$ *and* $c = 0$*, define*

$$\mu_t = \sum_{i=1}^{\infty} X_i(t) \delta_{t^{1/\beta} X_i(t)}.$$

Then, the random probability measures μ_t *converge in probability as* $t \to \infty$ *to a deterministic limit* μ_∞*, for the weak topology on measures. Furthermore, for* $k \in \mathbb{N}$*, we have*

$$\int_0^\infty y^{\beta k} \mu_\infty(\mathrm{d}y) = \frac{1}{\beta\, \Phi'(0+)} \prod_{i=1}^{k-1} \frac{i}{\Phi(i\beta)}. \tag{33}$$

Suppose the hypotheses of Lemma 6 are satisfied, and that $\beta > 0$ and $c = 0$. Let $\lambda(t)$ be a size-biased pick from the sequence $X(t) = (X_1(t), X_2(t), \dots)$. Note that μ_t is the conditional distribution of $t^{1/\beta}\lambda(t)$ given $X(t)$. The proof of the convergence in probability of μ_t to μ_∞ in [12] actually shows that for every f continuous and bounded, we have

$$\lim_{t\to\infty} E\left[\int_0^\infty f(y)\, \mu_t(\mathrm{d}y)\right] = \int_0^\infty f(y)\, \mu_\infty(\mathrm{d}y).$$

Therefore, the unconditional distributions γ_t of $t^{1/\beta}\lambda(t)$, given by

$$\gamma_t(B) = E[\mu_t(B)],$$

converge weakly to μ_∞ as $t \to \infty$.

6 One-dimensional distributions

Our goal in this section is to prove Theorem 2. The first step is Lemma 7. Once this lemma is proved, there is, for each $\alpha \in (0,1)$, only one remaining candidate for a self-similar fragmentation that could satisfy (2). To prove Theorem 2, we then only need to show that this fragmentation does indeed satisfy (2) only when $\alpha = 1/2$.

Lemma 7. *Fix* $\alpha \in (0,1)$. *Suppose* $(X(t), t \geqslant 0)$ *is a self-similar fragmentation with characteristics* (β, c, ν) *such that (2) holds. Then* $\beta = 1-\alpha$ *and* $c = 0$. *Also,* $(X(t), t \geqslant 0)$ *is binary, and the restriction* ν_1 *of* ν *to the first coordinate has density* $h_\alpha(x) = C_\alpha\, x^{-1-\alpha}(1-x)^{-1-\alpha}1_{[1/2,1]}(x)$.

Proof. Write the components of $X(t)$ as $(X_1(t), X_2(t), \dots)$. Let $J_1(t), J_2(t), \dots$ be the jump sizes of σ_α up to time t. Note that $\sum_{i=1}^\infty J_i(t) = \sigma_\alpha(t)$, so the conditional distribution of $\sum_{i=1}^\infty J_i(t)$ given $\sigma_\alpha(t) = 1$ is a unit mass at 1. Therefore, if (2) holds, we must have $\sum_{i=1}^\infty X_i(t) = 1$ almost surely. It follows from the construction in Sect. 2 that $c = 0$. Also, by section 3.3 of [12], we have $\beta \geqslant 0$.

Let $\lambda(t)$ be a size-biased pick from the sequence $(X_1(t), X_2(t), \dots)$. It follows from Lemma 2 that if (2) holds, then the distribution of $t^{1/(1-\alpha)}\lambda(t)$ converges to a nondegenerate limit. Combining this result with Lemma 6, we get $\beta = 1-\alpha$.

Suppose, for some $a > 0$, we have $\nu(\{x \in \Delta : x_1 + x_2 < 1-a\}) = b > 0$. Then, by Proposition 3 and the Portmanteau Theorem,

$$\liminf_{t\to 0} t^{-1} P\big(X_1(t) + X_2(t) < 1-a\big) \geqslant b.$$

Therefore, if (2) holds, then

$$\liminf_{t\to 0} t^{-1} P\big(J_1(t) + J_2(t) < 1-a\big) \geqslant b.$$

Let $J_1^*(t)$ be a size-biased pick from the jump sizes $J_1(t), J_2(t), \dots$, and let $J_2^*(t)$ be a size-biased pick from the remaining jump sizes. Note that $J_1^*(t) + J_2^*(t) \leqslant J_1(t) + J_2(t)$, so

$$\liminf_{t\to 0} t^{-1} P\big(J_1^*(t) + J_2^*(t) < 1-a\big) \geqslant b. \tag{34}$$

However, Lemma 4 implies that if $A = \{(x,y) \in [0,1]^2 : 0 < x+y < 1-a\}$, then

$$\lim_{t\to 0} t^{-2} P\big(J_1^*(t) + J_2^*(t) < 1-a\big) = \int_A \frac{C_\alpha^2\, x^{-\alpha} y^{-\alpha}(1-x-y)^{-1-\alpha}}{1-x}\, \mathrm{d}x\, \mathrm{d}y < \infty,$$

which contradicts (34). We conclude that $\nu(\{x \in \Delta : x_1 + x_2 < 1-a\}) = 0$ for all $a > 0$, which means X is a binary self-similar fragmentation.

Let μ_t be the measure on Δ defined by $\mu_t(A) = t^{-1}P(X(t) \in A)$. By Proposition 3, as $t \to 0$, μ_t converges weakly on complements of open neighborhoods of $(1, 0, \dots)$ to ν. Let $\tilde{\mu}_t$ be the measure defined by $\tilde{\mu}_t(B) = t^{-1}P(X_1(t) \in B)$. Let ν_1 be the restriction of ν to the first coordinate. Then $\tilde{\mu}_t$ converges weakly on $[0,a]$ to ν_1 as $t \to 0$ for any $a < 1$. It follows that

$$\lim_{t\to 0} t^{-1} P\big(X_1(t) \in [0,a]\big) = \lim_{t\to 0} \mu_t([0,a]) = \nu_1([0,a]) \tag{35}$$

for all $a \in [0,1)$ (the only interesting case is $a > 1/2$ since ν_1 does not support $[0,1/2]$) such that the function $x \mapsto \nu_1([0,x])$ is continuous at a. If (2) holds, then we can combine (35) with (17) to obtain

$$\int_0^a h_\alpha(x)\,\mathrm{d}x = \nu_1([0,a])$$

for all $a \in (1/2,1)$ such that $x \mapsto \nu_1([0,x])$ is continuous at a. Thus, h_α is the density of ν_1. $\square$

The binary self-similar fragmentation whose characteristics are $(1/2, 0, \nu)$, where the restriction of ν to the first coordinate has density $h_{1/2}$, is the Aldous–Pitman fragmentation. Therefore, Theorem 2 follows immediately from Lemma 7 and the following lemma.

Lemma 8. *Let $(X(t),\, t \geqslant 0)$ be a binary self-similar fragmentation with characteristics $(1-\alpha, 0, \nu)$, where the restriction of ν to the first coordinate has density h_α. If (2) holds, then $\alpha = 1/2$.*

Proof. Let $\lambda(t)$ be a size-biased pick from the sequence of $X(t)$. Let $\beta = 1-\alpha$. Let γ_t be the law of $t^{1/(1-\alpha)}\lambda(t) = t^{1/\beta}\lambda(t)$. Then, by results in Sect. 5, γ_t converges weakly to some measure μ_∞ as $t \to \infty$. Also, for all $k \in \mathbb{N}$, (33) gives

$$\int_0^\infty y^{\beta k}\,\mu_\infty(\mathrm{d}y) = \frac{1}{\beta\,\Phi'(0+)} \prod_{i=1}^{k-1} \frac{i}{\Phi(i\beta)},$$

where Φ is the function defined in (31).

Suppose (2) holds. By Lemma 2, μ_∞ is the Gamma$(1-\alpha,\ B\alpha/(1-\alpha))$ distribution. By (15),

$$\begin{aligned}\int_0^\infty y^{\beta k}\,\mu_\infty(\mathrm{d}y) &= \frac{\Gamma(\beta k + 1 - \alpha)}{\Gamma(1-\alpha)} \left(\frac{\cos(\pi\alpha/2)}{\alpha}\right)^{\beta k/(1-\alpha)} \\ &= \frac{\Gamma(\beta k + \beta)}{\Gamma(\beta)} \left(\frac{\cos(\pi\alpha/2)}{\alpha}\right)^{k}\end{aligned}$$

for all $k \in \mathbb{N}$. It follows that

$$\frac{1}{\beta\,\Phi'(0+)} \prod_{i=1}^{k-1} \frac{i}{\Phi(i\beta)} = \frac{\Gamma(\beta k + \beta)}{\Gamma(\beta)} \left(\frac{\cos(\pi\alpha/2)}{\alpha}\right)^{k} \tag{36}$$

for all $k \in \mathbb{N}$. By considering (36) for $k+1$ and k and taking the ratio of the two equations, we get

$$\frac{\Gamma(\beta k + 2\beta)}{\Gamma(\beta k + \beta)} \left(\frac{\cos(\pi\alpha/2)}{\alpha}\right) = \frac{k}{\Phi(k\beta)}. \tag{37}$$

Since $\alpha/\cos(\pi\alpha/2) = C_\alpha\,\Gamma(1-\alpha) = C_\alpha\,\Gamma(\beta)$ by (8), equation (37) implies

$$\Phi(k\beta) = \frac{C_\alpha \, k\Gamma(\beta)\Gamma(\beta k + \beta)}{\Gamma(\beta k + 2\beta)} \,. \tag{38}$$

By Stirling's Formula,

$$\lim_{k\to\infty} \frac{(k\beta)^\beta \Gamma(\beta k + \beta)}{\Gamma(\beta k + 2\beta)} = 1.$$

Combining this result with (38), we get

$$\lim_{k\to\infty} k^{\beta-1}\Phi(k\beta) = C_\alpha \, \Gamma(\beta)\beta^{-\beta}. \tag{39}$$

We will now compute $\lim_{k\to\infty} k^{\beta-1}\Phi(k\beta)$ directly from (31). We will show that the result agrees with the right-hand side of (39) only when $\beta = 1/2$, which will prove the lemma. Using the definitions of ν and h_α, equation (31), and the fact that $c = 0$, we have

$$\begin{aligned}\Phi(k\beta) &= \int_\Delta \left(1 - \sum_{i=1}^\infty x_i^{k\beta+1}\right) \nu(\mathrm{d}x) \\ &= C_\alpha \int_0^{1/2} \big(1 - x^{k\beta+1} - (1-x)^{k\beta+1}\big) x^{\beta-2}(1-x)^{\beta-2} \, \mathrm{d}x.\end{aligned}$$

By making the substitution $y = kx$, we get

$$\begin{aligned}k^{\beta-1}\Phi(k\beta) = C_\alpha \int_0^{k/2} &\big(1 - (k^{-1}y)^{k\beta+1} - (1-k^{-1}y)^{k\beta+1}\big) \\ &\times y^{\beta-2}(1-k^{-1}y)^{\beta-2} \, \mathrm{d}y.\end{aligned}$$

Note that for each fixed $y > 0$,

$$\begin{aligned}\lim_{k\to\infty} \big(1 - (k^{-1}y)^{k\beta+1} - (1-k^{-1}y)^{k\beta+1}\big) y^{\beta-2}(1-k^{-1}y)^{\beta-2} 1_{[0,k/2]}(y) \\ = (1 - \mathrm{e}^{-\beta y})y^{\beta-2}.\end{aligned}$$

If $0 \leqslant y \leqslant k/2$, then $(1-k^{-1}y)^{\beta-2} \leqslant 2^{2-\beta} \leqslant 4$. Also, if $0 < y < 1/2$, then $k \mapsto (1-k^{-1}y)^{k\beta+1}$ is an increasing function, and therefore $1 - (k^{-1}y)^{k\beta+1} - (1-k^{-1}y)^{k\beta+1} \leqslant 1 - (1-y)^{\beta+1} \leqslant 1 - (1-y)^2 \leqslant 2y$. Therefore, for all $k \in \mathbb{N}$,

$$\begin{aligned}&\big(1 - (k^{-1}y)^{k\beta+1} - (1-k^{-1}y)^{k\beta+1}\big) \\ &\qquad \times y^{\beta-2}(1-k^{-1}y)^{\beta-2} 1_{[0,k/2]}(y) \leqslant 4(2y \wedge 1)y^{\beta-2},\end{aligned}$$

and $\int_0^\infty 4(2y\wedge 1)y^{\beta-2} \, \mathrm{d}y < \infty$. Hence, by the Dominated Convergence Theorem,

$$\lim_{k\to\infty} k^{\beta-1}\Phi(k\beta) = C_\alpha \int_0^\infty (1 - \mathrm{e}^{-\beta y})y^{\beta-2} \, \mathrm{d}y.$$

Integrating by parts, we get

$$\lim_{k\to\infty} k^{\beta-1}\Phi(k\beta) = \frac{C_\alpha\,\beta}{1-\beta}\int_0^\infty y^{\beta-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y = \frac{C_\alpha\,\beta}{1-\beta}\,\Gamma(\beta)\beta^{-\beta}. \tag{40}$$

Combining (39) and (40), we get $\beta/(1-\beta)=1$, which means $\beta=1/2$ and therefore $\alpha=1/2$, as claimed. □

7 Mass of a tagged fragment

Our goal in this section is to prove Proposition 1, which pertains to the distribution of the asymptotic frequency of the block containing 1 in a partition-valued self-similar fragmentation or, equivalently, the distribution of a size-biased pick from a self-similar ranked fragmentation.

According to [11], the tagged fragment in a self-similar fragmentation with index β has to be the inverse of some increasing semi-stable Markov process of index $1/\beta$ started at 1. A semi-stable Markov process with index $1/\beta>0$ is a real-valued strong Markov process X satisfying the following self-similarity property. If, for $x>0$, P_x denotes the law of X starting from $X_0=x$, then for every $k>0$, the law of the process $(kX(k^{-\beta}s),\, s\geqslant 0)$ under P_x is the same as the law of $(X(s),\, s\geqslant 0)$ under P_{kx}.

Lemma 9. *Let $G(x,s)$ be a function defined on $[0,\infty)^2$ which is increasing in x and s. Suppose that there exists a semi-stable Markov process X with index $1/\beta$ such that $(G(x,\sigma_\alpha(t)),\, t\geqslant 0)$ has the law of X started at x. Then G is of the form*

$$G(x,s) = \left(x^{\beta/\alpha}+Ks\right)^{\alpha/\beta}$$

for some $K>0$.

Proof. By the scaling property, we have

$$\left(kG\left(x,\sigma_\alpha(k^{-\beta}t)\right),\, t\geqslant 0\right) =_d \left(kG\left(x,k^{-\beta/\alpha}\sigma_\alpha(t)\right),\, t\geqslant 0\right) \tag{41}$$

for all $k>0$. Since X is a semi-stable Markov process with index $1/\beta$, we have for all $k>0$,

$$\left(kG\left(x,\sigma_\alpha(k^{-\beta}t)\right),\, t\geqslant 0\right) =_d \left(G\left(kx,\sigma_\alpha(t)\right),\, t\geqslant 0\right). \tag{42}$$

Given k and x, define $f_1(s)=kG(x,k^{-\beta/\alpha}s)$ and $f_2(s)=G(kx,s)$. Then, f_1 and f_2 are increasing functions, and equations (41) and (42) imply that $f_1(\sigma_\alpha(t)) =_d f_2(\sigma_\alpha(t))$ for all $t>0$. For an increasing function f, define $f^{-1}(z)=\sup\{x : f(x)\leqslant z\}$. We have $P(f_1(\sigma_\alpha(t))\leqslant z)=P(f_2(\sigma_\alpha(t))\leqslant z)$, which means $P(\sigma_\alpha(t)\leqslant f_1^{-1}(z))=P(\sigma_\alpha(t)\leqslant f_2^{-1}(z))$. Since for all $t>0$ the density of $\sigma_\alpha(t)$ is positive on $(0,\infty)$, it follows that $f_1^{-1}(z)=f_2^{-1}(z)$ for all z. Therefore, if $f_1(s)<f_2(s)$, then $f_2(u)\leqslant f_1(s)$ for all $u<s$, and $f_1(u)\geqslant f_2(s)$ for all $u>s$. It follows that both f_1 and f_2 have a jump at s. Thus, $f_1(s)=f_2(s)$ for all but countably many s. Let $g(s)=G(1,s)$. Then, for all x, we have $G(x,s) = xg(x^{-\beta/\alpha}s)$ for all but countably many s. For any fixed $s > 0$, we have $P(\sigma_\alpha(t) \neq s \text{ for all } t) = 1$. Therefore, with probability one, $G(x, \sigma_\alpha(t)) = xg(x^{-\beta/\alpha}\sigma_\alpha(t))$ for all t. Thus, X under P_x has the same law as $(xg(x^{-\beta/\alpha}\sigma_\alpha(t)), t \geqslant 0)$ for some increasing function g.

By the Markov property,

$$P_y\big(X(t) \leqslant z\big) =_d P_x\big(X(s+t) \leqslant z \bigm| X(s) = y\big).$$

We have $P_y(X(t) \leqslant z) = P(\sigma_\alpha(t) \leqslant y^{\beta/\alpha} g^{-1}(z/y))$. Also,

$$\begin{aligned} P_x&\big(X(s+t) \leqslant z \bigm| X(s) = y\big) \\ &= P\big(\sigma_\alpha(s+t) \leqslant x^{\beta/\alpha} g^{-1}(z/x) \bigm| \sigma_\alpha(s) = x^{\beta/\alpha} g^{-1}(z/y)\big) \\ &= P\big(\sigma_\alpha(t) \leqslant x^{\beta/\alpha}\big(g^{-1}(z/x) - g^{-1}(y/x)\big)\big). \end{aligned}$$

It follows that for every $x \leqslant y \leqslant z$, we have

$$x^{\beta/\alpha}\left(g^{-1}\Big(\frac{z}{x}\Big) - g^{-1}\Big(\frac{y}{x}\Big)\right) = y^{\beta/\alpha} g^{-1}\Big(\frac{z}{y}\Big).$$

Writing $u = y/x$ and $v = z/y$ gives that for every $u, v \geqslant 1$, we have

$$g^{-1}(uv) = u^{\beta/\alpha} g^{-1}(v) + g^{-1}(u).$$

Taking $u = x$ and $v = 2$, we get $g^{-1}(2x) = x^{\beta/\alpha} g^{-1}(2) + g^{-1}(x)$. Taking $u = 2$ and $v = x$, we get $g^{-1}(2x) = 2^{\beta/\alpha} g^{-1}(x) + g^{-1}(2)$. It follows that $g^{-1}(x)(2^{\beta/\alpha} - 1) = g^{-1}(2)(x^{\beta/\alpha} - 1)$, which means $g^{-1}(x) = L(x^{\beta/\alpha} - 1)$ for some $L > 0$. Thus, $g(s) = G(1,s) = (1 + Ks)^{\alpha/\beta}$ for all s, where $K = L^{-1}$. It follows that for all x, we have $G(x,s) = xG(1, x^{-\beta/\alpha}(s)) = (x^{\beta/\alpha} + Ks)^{\alpha/\beta}$ for all but countably many s. Since G is increasing in x and s, we conclude that $G(x,s) = (x^{\beta/\alpha} + Ks)^{\alpha/\beta}$ for all x and s. □

Proof of Proposition 1. Suppose that $\lambda(t)$ is of the form $g(\sigma_\alpha(t))$ for some decreasing function g. By Lemma 9 and the preceding discussion, g must be of the form $g(x) = (1 + Kx)^{-\alpha/\beta}$ for some $\beta > 0$.

Set $h(x) = g^{-1}(x) = K^{-1}(x^{-\beta/\alpha} - 1)$. Then $h'(x) = -\beta K^{-1} x^{-(\beta/\alpha)-1}/\alpha$. Let f_t be the density of $\sigma_\alpha(t)$, and let $f = f_1$. Then the density of $\lambda(t)$ is given by

$$k(x) = f_t\big(h(x)\big)|h'(x)| = f\big(t^{-1/\alpha}K^{-1}(x^{-\beta/\alpha} - 1)\big)t^{-1/\alpha}\,\frac{\beta}{\alpha}\,K^{-1}x^{-(\beta/\alpha)-1}$$

for all $x \in (0,1)$. Note that

$$\big(t^{-1/\alpha}K^{-1}(x^{-\beta/\alpha} - 1)\big)^{-1-\alpha} t^{-1/\alpha} K^{-1} x^{(-\beta/\alpha)-1} = tK^\alpha x^{\beta-1}(1 - x^{\beta/\alpha})^{-1-\alpha}.$$

Therefore, it follows from (12) and the Dominated Convergence Theorem that if A is a Borel subset of $[a, 1-a]$ where $a > 0$, then

$$\lim_{t\to 0} t^{-1} P\big(\lambda(t) \in A\big) = \int_A a_1 \frac{K^\alpha \beta}{\alpha} x^{\beta-1}(1-x^{\beta/\alpha})^{-1-\alpha}\,\mathrm{d}x, \tag{43}$$

where a_1 is given in (11).

Let ν_i be the restriction of ν to the ith coordinate, and let γ be the measure defined by (23). By (43) and Corollary 1, γ is the measure on $[0,1]$ with density $a_1 K^\alpha \beta \alpha^{-1} x^{\beta-1}(1-x^{\beta/\alpha})^{-1-\alpha}$ for $x \in (0,1)$. Since Π is a binary fragmentation process,

$$\gamma(A) = \int_{A\cap[1/2,1]} x\,\nu_1(\mathrm{d}x) + \int_{A\cap[0,1/2]} x\,\nu_2(\mathrm{d}x).$$

Therefore, ν_1 has density $k_1(x) = a_1 K^\alpha \beta \alpha^{-1} x^{\beta-2}(1-x^{\beta/\alpha})^{-1-\alpha} 1_{[1/2,1]}(x)$, while ν_2 has density $k_2(x) = a_1 K^\alpha \beta \alpha^{-1} x^{\beta-2}(1-x^{\beta/\alpha})^{-1-\alpha} 1_{[0,1/2]}(x)$. However, since ν is concentrated on the set $\{x : x_1 + x_2 = 1\}$, we must have $k_1(x) = k_2(1-x)$ for all x. This gives that

$$\left(\frac{1-x^{\beta/\alpha}}{1-(1-x)^{\beta/\alpha}}\right)^{-1-\alpha} = \frac{(1-x)^{\beta-2}}{x^{\beta-2}}.$$

Comparing asymptotic behavior as $x \to 0$, we get $\beta = \alpha$ and then $\alpha = 1/2$. Note that $a_1 = (2\pi)^{-1/2} = C_{1/2}$ when $\alpha = 1/2$. Thus, $\nu_1(\mathrm{d}x) = (2\pi)^{-1/2} K^{1/2} x^{-3/2}(1-x)^{-3/2} 1_{[1/2,1]}(x)\,\mathrm{d}x$, which means that $(\Lambda(\Pi(t)), t \geqslant 0)$ is the Aldous–Pitman fragmentation up to a multiplicative time constant, as claimed. $\square$

Acknowledgments

The authors thank Jim Pitman for helpful discussions related to this work, and Jean Bertoin for his comments on a draft of this paper.

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