On the reduction of a multidimensional continuous martingale to a Brownian motion

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Knight's well-known theorem says that orthogonal continuous local martingales, when time-changed by their brackets, become independent Brownian motions (see [1], [7]–[11]). What can be said when the given local martingales are not orthogonal? The standard way to deal with this case is to orthogonalize them, for instance with the Gram–Schmidt algorithm. This is indeed what was done by Knight himself when first using his theorem (see [9], Theorem 2.2); but he was working in a particular setting (Hunt processes) and did not give explicit formulas. Other examples where this orthogonalization is used are references [3] and [12].

The goal of this short note is to provide expressions as explicit as possible to describe what is obtained when Knight's theorem is applied after orthogonalizing a family of continuous local martingales. Note that to orthogonalize the family of martingales we make use of some "local transformation" based on the matrix of predictable quadratic characteristics.

If A is a matrix, A' will denote the transpose of A. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ satisfying the usual conditions. We start by recalling Knight's theorem:

Theorem 1. *Let $M = (M(t))_{t\geqslant 0}$, $M(t) = (M_1(t), \dots, M_n(t))'$, be a n-dimensional continuous local martingale with orthogonal components, starting from zero. Suppose that on the same filtered probability space there exists a standard Brownian motion $\beta = (\beta(t))_{t\geqslant 0}$, $\beta(t) = (\beta_1(t), \dots, \beta_n(t))'$, starting from zero and independent of M.*

Then the process $B = (B(t))_{t\geqslant 0}$, $B(t) = (B_1(t), \dots, B_n(t))'$,

$$B_k(t) = \begin{cases} M_k(\tau_t^k), & \text{if } \langle M_k, M_k\rangle(\infty) > t, \\ M_k(\infty) + \beta_k(t - \langle M_k, M_k\rangle(\infty)), & \text{if } \langle M_k, M_k\rangle(\infty) < t, \end{cases} \tag{1}$$

is a Brownian motion, where

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$$\tau_t^k = \inf\{s : \langle M_k, M_k\rangle(s) > t\}.$$

Remark 1. The one-dimensional version of this result was proved in [2], [4].

Remark 2. The processes B_k and B are Brownian motions relative to their natural filtrations $\big(\mathcal{F}_t^{B_k}\big)_{t\geqslant 0}$ and $\big(\bigvee_i \mathcal{F}_t^{B_i}\big)_{t\geqslant 0}$ respectively.

Now we are given $M = (M(t))_{t\geqslant 0}$, $M(t) = (M_1(t), \dots, M_n(t))'$, a column-vector of continuous local martingales starting from zero.

Denote by

$$\langle M\rangle(t) = \big(\langle M_i, M_j\rangle(t)\big)_{1\leqslant i,j\leqslant n}$$

the matrix of predictable quadratic characteristics of M (see [5], [6]), and set

$$C(t) = \left(\frac{\mathrm{d}\langle M_i, M_j\rangle}{\mathrm{d}a}(t)\right)_{1\leqslant i,j\leqslant n}, \tag{2}$$

where

$$a(t) = \mathrm{tr}\langle M\rangle(t) = \sum_{i=1}^{n}\langle M_i, M_i\rangle(t).$$

The matrix C is predictable symmetric non-negative. There exists a predictable orthogonal matrix T and a predictable diagonal matrix D such that

$$T'CT = D = (d_i)_{1\leqslant i\leqslant n}, \tag{3}$$

where all $d_i \geqslant 0$, Q-a.s., where Q is the measure on the predictable σ-field such that $\mathrm{d}Q = \mathrm{d}a \times \mathrm{d}\mathbb{P}$. The matrix T can be chosen predictable because its columns are the orthonormal basis of eigenvectors of C; and $d_i, i = 1, \dots, n$, are the eigenvalues of C.

Theorem 2. *Let $M = (M(t))_{t\geqslant 0}$, $M(t) = (M_1(t), \dots, M_n(t))'$, be a n-dimensional continuous local martingale starting from zero, with matrix C of predictable local quadratic characteristics (see (2)). Suppose that on the same filtered probability space there exists a standard Brownian motion $\beta = (\beta(t))_{t\geqslant 0}$, $\beta(t) = (\beta_1(t), \dots, \beta_n(t))'$, starting from zero and independent of M. Then*

(i) *the process $X = (X(t))_{t\geqslant 0}$, $X(t) = (X_1(t), \dots, X_n(t))'$, given by*

$$X(t) = \int_0^t T'(s)\,\mathrm{d}M(s) \tag{4}$$

is a n-dimensional continuous local martingale with orthogonal components; the matrix of predictable quadratic characteristics of X equals

$$\langle X\rangle(t) = \int_0^t D(s)\,\mathrm{d}a(s) = \left(\int_0^t d_i(s)\,\mathrm{d}a(s)\right)_{1\leqslant i\leqslant n}; \tag{5}$$

(ii) *the process* $B = (B(t))_{t \geqslant 0}$, $B(t) = (B_1(t), \ldots, B_n(t))'$, *is a* n*-dimensional Brownian motion, where*

$$B_k(t) = \begin{cases} X_k(\tau_t^k), & \text{if } \langle X_k, X_k\rangle(\infty) > t, \\ X_k(\infty) + \beta_k(t - \langle X_k, X_k\rangle(\infty)), & \text{if } \langle X_k, X_k\rangle(\infty) < t, \end{cases} \tag{6}$$

$$\langle X_k, X_k\rangle(t) = \int_0^t d_k(s)\,\mathrm{d}a(s),$$

$$\tau_t^k = \inf\{s : \langle X_k, X_k\rangle(s) > t\}.$$

Proof. (i) The equality $\mathbb{E}[X(T)X'(T)] = \mathbb{E}[\langle X\rangle(T)]$, valid for any bounded stopping time T, and (3) imply that

$$\begin{aligned}\langle X\rangle(t) = \big(\langle X_i, X_j\rangle(t)\big)_{1 \leqslant i,j \leqslant n} &= \int_0^t T'(s)\,\mathrm{d}\langle M\rangle(s) T(s) \\ &= \int_0^t T'CT(s)\,\mathrm{d}a(s) = \int_0^t D(s)\,\mathrm{d}a(s) \\ &= \left(\int_0^t d_i(s)\,\mathrm{d}a(s)\right)_{1 \leqslant i \leqslant n}.\end{aligned}$$

Hence (5). Since the matrix $\langle X\rangle$ is diagonal, the components of the martingale X are orthogonal.

The assertion (ii) follows from (i) and Theorem 1. □

Remark 3. Formula (4) defines a "local transformation" of the martingale M to a martingale with orthogonal components.

Remark 4. Relations (4), (6) imply that

$$M(t) = \int_0^t T(s)\,\mathrm{d}B \circ \langle M, M\rangle(s), \qquad t \geqslant 0,$$

where

$$B \circ \langle M, M\rangle(s) = \Big(B_1\big(\langle M_1, M_1\rangle(s)\big), \ldots, B_n\big(\langle M_n, M_n\rangle(s)\big)\Big)'.$$

Remark 5. An original extension of Knight's theorem for a finite or countable family of continuous local martingales $M_1, M_2, \ldots$ such that $\langle M_i, M_j\rangle = 0$, for all $i \neq j$, is given by Kallenberg ([7], Proposition 16.8). He uses an isometry between Gaussian processes and some continuous martingales to obtain the independence of processes like $B_1, B_2, \ldots$ in (6); this provides a new proof of Knight's theorem, in a coordinate-free framework. Using this, Theorem 2 can be extended to the case of Hilbert-valued continuous local martingales.

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