The Codimension of the Zeros of a Stable Process in Random Scenery

Davar Khoshnevisan*

The University of Utah, Department of Mathematics
Salt Lake City, UT 84105–0090, U.S.A.
e-mail: davar@math.utah.edu
http://www.math.utah.edu/~davar

Summary. We show that for any $\alpha \in (1, 2]$, the (stochastic) codimension of the zeros of an α-stable process in random scenery is identically $1 - (2\alpha)^{-1}$. As an immediate consequence, we deduce that the Hausdorff dimension of the zeros of the latter process is almost surely equal to $(2\alpha)^{-1}$. This solves Conjecture 5.2 of [6], thereby refining a computation of [10].

Key words: Random walk in random scenery; stochastic codimension; Hausdorff dimension.

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Introduction

A stable process in random scenery is the continuum limit of a class of random walks in random scenery that is described as follows. A *random scenery* on $\mathbb{Z}$ is a collection, $\{y(0), y(\pm 1), y(\pm 2), \dots\}$, of i.i.d. mean-zero variance-one random variables. Given a collection $x = \{x_1, x_2, \dots\}$ of i.i.d. random variables, we consider the usual random walk $n \mapsto s_n = x_1 + \cdots + x_n$ which leads to the following *random walk in random scenery*:

$$w_n = y(s_1) + \cdots + y(s_n), \qquad n = 1, 2, \dots \tag{1}$$

In words, w is obtained by summing up the values of the scenery that are encountered by the ordinary random walk s.

Consider the *local times* $\{l_n^a;\, a \in \mathbb{Z}, n = 1, 2, \dots\}$ of the ordinary random walk s:

$$l_n^a = \sum_{j=1}^{n} \mathbf{1}_{\{a\}}(s_j), \qquad a \in \mathbb{Z},\ n = 1, 2, \dots$$

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Then, one readily sees from (1) that

$$w_n = \sum_{a \in \mathbb{Z}} l_n^a y(a), \qquad n = 1, 2, \ldots$$

As soon as s is in the domain of attraction of a stable process of index $\alpha \in (1, 2]$, one might expect its local times to approximate those of the limiting stable process. Thus, one may surmise an explicit weak limit for a renormalization of w. Reference [4] has shown that this is the case. Indeed, let $S = \{S(t);\, t \geqslant 0\}$ denote a stable Lévy process with Lévy exponent

$$\mathbb{E}\big[\exp\big(\mathrm{i}\xi S(1)\big)\big] = \exp\bigg(-|\xi|^\alpha \frac{1 + \mathrm{i}\nu \operatorname{sgn}(\xi) \tan(\alpha\pi/2)}{\chi}\bigg), \qquad \xi \in \mathbb{R}, \tag{2}$$

where $\nu \in [-1, 1]$ and $\chi > 0$. If $\alpha \in (1, 2]$, then it is well known ([1]) that S has continuous local times; i.e., there exists a continuous process $(x, t) \mapsto L_t(x)$ such that for all Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$, and all $t \geqslant 0$,

$$\int_0^t f\big(S(u)\big)\, du = \int_{-\infty}^{\infty} f(a) L_t(a)\, \mathrm{d}a. \tag{3}$$

Then, according to [4], as long as s is in the domain of attraction of S, the random walk in random scenery w can be normalized to converge weakly to the *stable process in random scenery* W defined by

$$W(t) = \int_{-\infty}^{\infty} L_t(x)\, B(\mathrm{d}x). \tag{4}$$

Here $B = \{B(t);\, -\infty < t < +\infty\}$ is a two-sided Brownian motion that is totally independent of the process S, and the stochastic integral above is defined in the sense of N. Wiener or, more generally, K. Itô.

References [5, 6] have established a weak notion of duality between iterated Brownian motion (i.e., $B \circ B'$, where B' is an independent Brownian motion) and Brownian motion in random scenery (i.e., the process W when $\alpha = 2$). Since the level sets of iterated Brownian motion have Hausdorff dimension $3/4$ ([2]), this duality suggests that when $\alpha = 2$ the level sets of W ought to have Hausdorff dimension $1/4$; cf. [6, Conjecture 5.2]. Reference [10] has shown that a randomized version of this assertion is true: For the $\alpha = 2$ case, and for any $t > 0$,

$$\mathbb{P}\Big\{\dim\big(W^{-1}\{W(t)\}\big) = \frac{1}{4}\Big\} = 1,$$

where $W^{-1}A = \{s \geqslant 0 :\, W(s) \in A\}$ for any Borel set $A \subset \mathbb{R}$. In particular, Lebesgue-almost all level sets of W have Hausdorff dimension $1/4$ when $\alpha = 2$.

In this note, we propose to show that the preceeding conjecture is true for all level sets, and has an extension for all $\alpha \in (1, 2]$. Indeed, we offer the following stronger theorem whose terminology will be explained shortly.

Theorem 1. *For any $x \in \mathbb{R}$,*

$$\operatorname{codim}\big(W^{-1}\{x\}\big) = 1 - \frac{1}{2\alpha}. \tag{5}$$

Consequently, if dim *represents Hausdorff dimension, then*

$$\dim\big(W^{-1}\{x\}\big) = \frac{1}{2\alpha}, \qquad \textit{almost surely.} \tag{6}$$

To conclude the introduction, we will define stochastic codimension, following the treatment of [8].

Given a random subset, K, of $\mathbb{R}_+$, we can define the *lower* (*upper*) *stochastic codimension* of K as the largest (smallest) number β such that for all compact sets $F \subset \mathbb{R}_+$ whose Hausdorff dimension is strictly less (greater) than β, K cannot (can) intersect F. We write $\underline{\operatorname{codim}}(K)$ and $\overline{\operatorname{codim}}(K)$ for the lower and the upper stochastic codimensions of K, respectively. When they agree, we write $\operatorname{codim}(K)$ for their common value, and call it the (stochastic) *codimension* of K. Note that the upper and the lower stochastic codimensions of K are not random, although K is a random set.

1 Supporting Lemmas

We recall from [8, Theorem 2.2] and its proof that when a random set $X \subseteq \mathbb{R}$ has a stochastic codimension,

$$\operatorname{codim} X + \dim X = 1, \qquad \mathbb{P}\text{-a.s.}$$

This shows that (5) implies (6). Thus, we will only verify (5). Throughout, P (resp. E) denote the conditional probability measure (resp. expectation) $\mathbb{P}$ (resp. $\mathbb{E}$), given the entire process S.

With the above notation in mind, it should be recognized that, under the measure P, the process W is a centered Gaussian process with covariance

$$E[W(s)W(t)] = \langle L_s, L_t \rangle, \qquad s, t \geqslant 0, \tag{7}$$

where $\langle \,.\,,\,.\, \rangle$ denotes the usual $\mathcal{L}^2(\mathbb{R})$-inner product. Needless to say, the above equality holds with $\mathbb{P}$-probability one. In particular, $\mathbb{P}$-a.s., the P-variance of $W(t)$ is $\|L_t\|_2^2$, where $\|\,.\,\|_r$ denotes the usual $\mathcal{L}^r(\mathbb{R})$-norm for any $1 \leqslant r \leqslant \infty$.

By the Cauchy–Schwarz inequality, $\langle f, g \rangle^2 \leqslant \|f\|_2^2 \cdot \|g\|_2^2$. We need the following elementary estimate for the slack in this inequality. It will translate to a P-correlation estimate for the process W.

Lemma 1. *For all $f, g \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$,*

$$\|f\|_2^2\,\|g\|_2^2 - \langle f, g \rangle^2 \geqslant \|g\|_2^2\,\|f-g\|_2^2 - \|g\|_\infty^2\,\|f-g\|_1^2.$$

Proof. One can check the following variant of the parallelogram law on $\mathcal{L}^2(\mathbb{R})$:

$$\|f\|_2^2\,\|g\|_2^2 - \langle f,g\rangle^2 = \|f-g\|_2^2\,\|g\|_2^2 - \langle f-g,g\rangle^2,$$

from which the lemma follows immediately. □

Now, consider the random field

$$\varrho_{s,t} = \frac{\langle L_s, L_t\rangle}{\|L_s\|_2^2}, \qquad s,t \geqslant 0. \tag{8}$$

Under the measure P, $\{\varrho_{s,t};\ s,t \geqslant 0\}$ can be thought of as a collection of constants. Then, one has the following conditional regression bound:

Lemma 2 (Conditional Regression). *Fix* $1 \leqslant s < t \leqslant 2$. *Then, under the measure* P, $W(s)$ *is independent of* $W(t) - \varrho_{s,t}W(s)$. *Moreover,* $\mathbb{P}$*-a.s.,*

$$E\Big[\big|W(t)-\varrho_{s,t}W(s)\big|^2\Big] \geqslant \left(\|L_t - L_s\|_2^2 - \frac{\|L_2\|_\infty^2}{\|L_1\|_2^2}|t-s|^2\right)_+. \tag{9}$$

Proof. The independence assertion is an elementary result in linear regression. Indeed, it follows from the conditional Gaussian distribution of the process W, together with the following consequence of (7):

$$E\big[W(s)\big(W(t)-\varrho_{s,t}W(s)\big)\big] = 0, \qquad \mathbb{P}\text{-a.s.}$$

Similarly, (conditional) regression shows that $\mathbb{P}$-a.s.,

$$E\Big[\big(W(t)-\varrho_{s,t}W(s)\big)^2\Big] = \frac{\|L_t\|_2^2\,\|L_s\|_2^2 - \langle L_s, L_t\rangle^2}{\|L_s\|_2^2}, \tag{10}$$

$\mathbb{P}$-a.s. Thanks to Lemma 1, the numerator is bounded below by

$$\|L_s\|_2^2\,\|L_t - L_s\|_2^2 - \|L_s\|_\infty^2\,\|L_t - L_s\|_1^2.$$

By the occupation density formula (3), with $\mathbb{P}$-probability one, $\|L_t - L_s\|_1 = (t-s)$. Since $r \mapsto L_r(x)$ is non-increasing for any $x \in \mathbb{R}$, the lemma follows from (10). □

Now, we work toward showing that the right hand side of (9) is essentially equal to the much simpler expression $\|L_t - L_s\|_2^2$. This will be done in a few steps.

Lemma 3. *If* $0 \leqslant s < t$ *are fixed, then the* $\mathbb{P}$*-distribution of* $\|L_t - L_s\|_2^2$ *is the same as that of* $(t-s)^{2-(1/\alpha)}\|L_1\|_2^2$.

Proof. Since the stable process S is Lévy, by applying the Markov property at time t, we see that the process $L_t(\,.\,) - L_s(\,.\,)$ has the same finite dimensional distributions as $L_{t-s}(\,.\,)$. The remainder of this lemma follows from scaling; see [7, 5.4], for instance. □

Next, we introduce a somewhat rough estimate of the modulus of continuity of the infinite-dimensional process $t \mapsto L_t$.

Lemma 4. *For each $\eta > 0$, there exists a $\mathbb{P}$-a.s. finite random variable V_4 such that for all $0 \leqslant s < t \leqslant 2$,*

$$\|L_t - L_s\|_2^2 \leqslant V_4 |t-s|^{2-(1/\alpha)-\eta}.$$

Proof. Thanks to Lemma 3, for any $\nu > 1$, and for all $0 \leqslant s < t$,

$$\mathbb{E}\big[\|L_t - L_s\|_2^{2\nu}\big] = (t-s)^{2\nu-(\nu/\alpha)}\, \mathbb{E}\big[\|L_1\|_2^{2\nu}\big].$$

On the other hand, by the occupation density formula (3),

$$\|L_1\|_2^2 \leqslant \|L_1\|_\infty \int_{-\infty}^{\infty} L_1(x)\,\mathrm{d}x = \|L_1\|_\infty.$$

According to [9, Theorem 1.4], there exists a finite $c > 0$ such that

$$\mathbb{P}\{\|L_1\|_\infty > \lambda\} \leqslant \exp(-c\lambda^\alpha), \qquad \forall \lambda > 1.$$

The result follows from the preceeding two displays, used in conjunction with Kolmogorov's continuity criterion applied to the $\mathcal{L}^2(\mathbb{R})$-valued process $t \mapsto L_t$. □

Up to an infinitesimal in the exponent, the above is sharp, as the following asserts.

Lemma 5. *For each $\eta > 0$, there exists a $\mathbb{P}$-a.s. finite random variable V_5 such that for all $1 \leqslant s < t \leqslant 2$,*

$$\|L_t - L_s\|_2^2 \geqslant V_5 |t-s|^{2-(1/\alpha)+\eta}.$$

Proof. According to [7, proof of Lemma 5.4], there exists a finite constant $c > 0$ such that for all $\lambda \in (0,1)$,

$$\mathbb{P}\big\{\|L_1\|_2^2 \leqslant \lambda\big\} \leqslant \exp(-c\lambda^{-\alpha}). \tag{11}$$

Combined with Lemma (3), this yields

$$\mathbb{P}\big\{\|L_{s+h} - L_s\|_2^2 \leqslant h^{2-(1/\alpha)+\eta}\big\} \leqslant \exp(-ch^{-\eta}), \qquad s \in [1,2],\, h \in (0,1).$$

Let

$$F_n = \big\{k2^{-n};\, 0 \leqslant k \leqslant 2^{n+1}\big\}, \qquad n = 0, 1, \ldots$$

Choose and fix some number $p > \eta^{-1}$ to see that

$$\mathbb{P}\Big\{\min_{s \in F_n} \|L_{s+n^{-p}} - L_s\|_2^2 \leqslant n^{-p\gamma}\Big\} \leqslant (2^{n+1}+1)\exp(-cn^{\eta p}),$$

where $\gamma = 2 - (1/\alpha) + \eta$. Since $p > \eta^{-1}$, the above probability sums in n. By the Borel–Cantelli lemma, $\mathbb{P}$-almost surely,

$$\min_{s \in F_n} \|L_{s+n^{-p}} - L_s\|_2^2 \geqslant n^{-p\gamma}, \qquad \text{eventually.} \tag{12}$$

On the other hand, any for any $s \in [1,2]$, there exists $s' \in F_n$ such that $|s - s'| \leqslant 2^{-n}$. In particular,

$$\inf_{s \in [1,2]} \|L_{s+n^{-p}} - L_s\|_2^2 \geqslant \min_{s \in F_n} \|L_{s+n^{-p}} - L_s\|_2^2 - 4 \sup_{\substack{0 \leqslant u,v \leqslant 2 \\ |u-v| \leqslant 2^{-n}}} \|L_u - L_v\|_2^2.$$

We have used the inequality $|x+y|^2 \leqslant 2(x^2+y^2)$ to obtain the above. Thus, by Lemma 4, and by (12), $\mathbb{P}$-almost surely,

$$\inf_{s \in [1,2]} \|L_{s+n^{-p}} - L_s\|_2^2 \geqslant (1+o(1))n^{-p\gamma}, \qquad \text{eventually.}$$

Since $t \mapsto L_t(x)$ is increasing, the preceeding display implies the lemma. $\square$

2 Proof of Theorem 1

Not surprisingly, we prove Theorem 1 in two steps: First, we obtain a lower bound for $\text{codim}(W^{-1}\{x\})$. Then, we establish a corresponding upper bound.

In order to simplify the notation, we only work with the case $x = 0$; the general case follows by the very same methods.

2.1 The Lower Bound

The lower bound is quite simple, and follows readily from Lemma 4 and the following general result.

Lemma 6. *If $\{Z(t);\, t \in [1,2]\}$ is almost surely Hölder continuous of some non-random order $\gamma > 0$, and if $Z(t)$ has a bounded density function uniformly for every $t \in [1,2]$, then*

$$\underline{\text{codim}}(Z^{-1}\{0\}) \geqslant \gamma.$$

Proof. If $F \subset \mathbb{R}_+$ is a compact set whose Hausdorff dimension is $< \gamma$, then we are to show that almost surely, $Z^{-1}\{0\} \cap F = \varnothing$.

By the definition of Hausdorff dimension, and since $\dim(F) < \gamma$, for any $\delta > 0$ we can find closed intervals $F_1, F_2, \ldots$ such that

$$\text{(i)} \quad F \subseteq \cup_{i=1}^{\infty} F_i \qquad \text{and} \qquad \text{(ii)} \quad \sum_{i=1}^{\infty} (\text{diam}\, F_i)^{\gamma} \leqslant \delta.$$

Let s_i denote the left endpoint of F_i, and observe that whenever $Z^{-1}\{0\}\cap F_j \neq \varnothing$, then with $\mathbb{P}$-probability one,

$$|Z(s_j)| \leqslant \sup_{s,t\in F_j} |Z(s) - Z(t)| \leqslant K_\gamma (\operatorname{diam} F_j)^\gamma,$$

where K_γ is an almost surely finite random variable that signifies the Hölder constant of Z. In particular, for any $M > 0$,

$$\begin{aligned}\mathbb{P}\big\{Z^{-1}\{0\} \cap F \neq \varnothing\big\} &\leqslant \sum_{j=1}^{\infty} \mathbb{P}\big\{|Z(s_j)| \leqslant M(\operatorname{diam} F_j)^\gamma\big\} + \mathbb{P}\{K_\gamma > M\} \\ &\leqslant 2DM \sum_{j=1}^{\infty} (\operatorname{diam} F_j)^\gamma + \mathbb{P}\{K_\gamma > M\},\end{aligned}$$

where D is the uniform bound on the density function of $Z(t)$, as t varies in $[1, 2]$. Consequently,

$$\mathbb{P}\big\{Z^{-1}\{0\} \cap F \neq \varnothing\big\} \leqslant 2DM\delta + \mathbb{P}\{K_\gamma > M\}.$$

Since δ is arbitrary,

$$\mathbb{P}\big\{Z^{-1}\{0\} \cap F \neq \varnothing\big\} \leqslant \mathbb{P}\{K_\gamma > M\},$$

which goes to zero as $M \to \infty$. □

We can now turn to our

Proof of Theorem 1: Lower Bound. Since W is Gaussian under the measure P, for any $\nu > 0$, there exists a non-random and finite constant $C_\nu > 0$ such that for all $0 \leqslant s \leqslant t \leqslant 2$,

$$\begin{aligned}E\big[|W(s) - W(t)|^\nu\big] &= C_\nu \Big(E\big[|W(s) - W(t)|^2\big]\Big)^{\nu/2} \\ &= C_\nu \|L_t - L_s\|_2^\nu.\end{aligned}$$

Taking $\mathbb{P}$-expectations and appealing to Lemma 3 leads to

$$\mathbb{E}\big[|W(s) - W(t)|^\nu\big] = C'_\nu (t - s)^{\nu - (\nu/2\alpha)},$$

where $C'_\nu = C_\nu \mathbb{E}[\|L_1\|_2^\nu]$ is finite, thanks to [9, Theorem 1.4]. By Kolmogorov's continuity theorem, with probability one, $t \mapsto W(t)$ is Hölder continuous of any order $\gamma < 1 - (2\alpha)^{-1}$. We propose to show that the density function of $W(t)$ is bounded uniformly for all $t \in [1, 2]$. Lemma 6 would then show that $\operatorname{codim}(W^{-1}\{0\} \cap [1, 2]) \geqslant \gamma$ for any $\gamma < 1 - (2\alpha)^{-1}$; i.e., $\operatorname{codim}(W^{-1}\{0\} \cap [1, 2]) \geqslant 1 - (2\alpha)^{-1}$. The argument to show this readily implies that $\operatorname{codim}(W^{-1}\{0\}) \geqslant 1 - (2\alpha)^{-1}$, which is the desired lower bound.

To prove the uniform boundedness assertion on the density function, f_t, of $W(t)$, we condition first on the entire process S to obtain

$$f_t(x) = \frac{1}{\sqrt{2\pi}} \mathbb{E}\left[\frac{1}{\|L_t\|_2} \exp\left(-\frac{x^2}{2\|L_t\|_2^2} \right) \right], \qquad t \in [1,2],\ x \in \mathbb{R}.$$

In particular,

$$\sup_{t\in[1,2]} \sup_{x\in\mathbb{R}} f_t(x) \leqslant \mathbb{E}\left[\|L_1\|_2^{-1} \right],$$

which is finite, thanks to (11). □

2.2 The Upper Bound

We intend to show that for any $x \in \mathbb{R}$, and for any compact set $F \subset \mathbb{R}_+$ whose Hausdorff dimension is $> 1 - (2\alpha)^{-1}$, $\mathbb{P}\{W^{-1}\{x\} \cap F \neq \varnothing\} > 0$. It suffices to show that for all such F's,

$$P\left\{ W^{-1}\{x\} \cap F \neq \varnothing \right\} > 0, \qquad \mathbb{P}\text{-a.s.}$$

As in our lower bound argument, we do this merely for $x = 0$ and $F \subseteq [1,2]$, since the general case is not much different. Henceforth, we shall fix one such compact set F without further mention.

Let $\mathcal{P}(F)$ denote the collection of probability measures on F, and for all $\mu \in \mathcal{P}(F)$ and all $\varepsilon > 0$, define

$$J_\varepsilon(\mu) = \frac{1}{2\varepsilon} \int \mathbf{1}_{\{|W(s)| \leqslant \varepsilon\}}\, \mu(\mathrm{d}s). \tag{13}$$

We proceed to estimate the first two moments of $J_\varepsilon(\mu)$.

Lemma 7. *There exists a $\mathbb{P}$-a.s. finite and positive random variable V_7 such that $\mathbb{P}$-almost surely,*

$$\liminf_{\varepsilon \to 0} E[J_\varepsilon(\mu)] \geqslant V_7,$$

for any $\mu \in \mathcal{P}(F)$.

Proof. Notice the explicit calculation:

$$E[J_\varepsilon(\mu)] = \frac{1}{2\sqrt{2\pi}\varepsilon} \int_F \int_{-\varepsilon}^{+\varepsilon} \|L_s\|_2^{-1} \exp\left(-\frac{x^2}{2\|L_s\|_2^2} \right) \mathrm{d}x\, \mu(\mathrm{d}s),$$

valid for all $\varepsilon > 0$ and all $\mu \in \mathcal{P}(F)$. Since $F \subseteq [1,2]$, the monotonicity of local times shows that

$$E[J_\varepsilon(\mu)] \geqslant \frac{1}{\sqrt{2\pi}} \|L_2\|_2^{-1} \exp\left(-\frac{\varepsilon^2}{2\|L_1\|_2^2} \right).$$

The lemma follows with $V_7 = (2\pi)^{-\frac{1}{2}} \|L_2\|_2^{-1}$, which is $\mathbb{P}$-almost surely (strictly) positive, thanks to (11). □

Lemma 8. *For any $\eta > 0$, there exists a $\mathbb{P}$-a.s. positive and finite random variable V_8 such that for all $\mu \in \mathcal{P}(F)$,*

$$\sup_{\varepsilon \in (0,1)} E\big[|J_\varepsilon(\mu)|^2\big] \leqslant V_8 \iint |s-t|^{-1+(1/2\alpha)-\eta}\,\mu(\mathrm{d}s)\,\mu(\mathrm{d}t), \qquad \mathbb{P}\text{-}a.s.$$

Proof. We recall $\varrho_{s,t}$ from (8), and observe that for any $1 \leqslant s < t \leqslant 2$, and for all $\varepsilon > 0$,

$$\begin{aligned} P\{|W(s)| \leqslant \varepsilon,\, |W(t)| \leqslant \varepsilon\} \\ &= P\{|W(s)| \leqslant \varepsilon,\, |W(t) - \varrho_{s,t}W(s) + \varrho_{s,t}W(s)| \leqslant \varepsilon\} \\ &\leqslant P\{|W(s)| \leqslant \varepsilon\} \times \sup_{x \in \mathbb{R}} P\{|W(t) - \varrho_{s,t}W(s) + x| \leqslant \varepsilon\}, \end{aligned}$$

since $W(s)$ and $W(t) - \varrho_{s,t}W(s)$ are P-independent; cf. Lemma 2. On the other hand, centered Gaussian laws are unimodal. Hence, the above supremum is achieved at $x = 0$. That is,

$$P\{|W(s)| \leqslant \varepsilon,\, |W(t)| \leqslant \varepsilon\} \leqslant P\{|W(s)| \leqslant \varepsilon\} \times P\{|W(t) - \varrho_{s,t}W(s)| \leqslant \varepsilon\}.$$

Computing explicitly, we obtain

$$\sup_{s \in [1,2]} P\{|W(s)| \leqslant \varepsilon\} \leqslant \varepsilon \|L_1\|_2^{-1}. \tag{14}$$

Likewise,

$$P\{|W(t) - \varrho_{s,t}W(s)| \leqslant \varepsilon\} \leqslant \frac{\varepsilon}{\sqrt{E[|W(t) - \varrho_{s,t}W(s)|^2]}}, \qquad \mathbb{P}\text{-a.s.}$$

We can combine (14) with conditional regression (Lemma 2) and Lemma 5, after a few lines of elementary calculations. ☐

Proof of Theorem 1: Upper Bound. Given a compact set $F \subset [1,2]$ with $\dim(F) > 1 - (2\alpha)^{-1}$, we are to show that $P\{W^{-1}\{0\} \cap F \neq \varnothing\} > 0$, $\mathbb{P}$-almost surely. But, for any $\mu \in \mathcal{P}(F)$, the following holds $\mathbb{P}$-a.s.:

$$\begin{aligned} P\{W^{-1}\{0\} \cap F \neq \varnothing\} &\geqslant \liminf_{\varepsilon \to 0} P\{J_\varepsilon(\mu) > 0\} \\ &\geqslant \liminf_{\varepsilon \to 0} \frac{\big|E[J_\varepsilon(\mu)]\big|^2}{E[|J_\varepsilon(\mu)|^2]}, \end{aligned}$$

thanks to the classical Paley–Zygmund inequality ([3, p. 8]). Lemmas 7 and 8, together imply that for any $\eta > 0$, $\mathbb{P}$-almost surely,

$$P\{W^{-1}\{0\} \cap F \neq \varnothing\} \geqslant \frac{V_7^2}{V_8 \times \inf_{\mu \in \mathcal{P}(F)} \iint |s-t|^{-1+(1/2\alpha)-\eta}\,\mu(\mathrm{d}s)\,\mu(\mathrm{d}t)}.$$

Note that the random variable V_8 depends on the value of $\eta > 0$. Now, choose η such that $\dim(F) > 1 - (2\alpha)^{-1} + \eta$, and apply Frostman's theorem ([3, p. 130]) to deduce that

$$\inf_{\mu \in \mathcal{P}(F)} \iint |s-t|^{-1+(1/2\alpha)-\eta}\, \mu(\mathrm{d}s)\, \mu(\mathrm{d}t) < +\infty.$$

This concludes our proof. □

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