A note on representations of eigenvalues of classical Gaussian matrices

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Summary. We use a matrix central-limit theorem which makes the Gaussian Unitary Ensemble appear as a limit of the Laguerre Unitary Ensemble together with an observation due to Johansson in order to derive new representations for the eigenvalues of GUE. For instance, it is possible to recover the celebrated equality in distribution between the maximal eigenvalue of GUE and a last-passage time in some directed Brownian percolation. Similar identities for the other eigenvalues of GUE also appear.

1 Introduction

The most famous ensembles of Hermitian random matrices are undoubtedly the Gaussian Unitary Ensemble (GUE) and the Laguerre Unitary Ensemble (LUE). Let $(X_{i,j})_{1\leqslant i<j\leqslant N}$ (respectively $(X_{i,i})_{1\leqslant i\leqslant N}$) be complex (respectively real) standard independent Gaussian variables ($\mathbb{E}(X_{i,j})=0$, $\mathbb{E}(|X_{i,j}|^2)=1$) and let $X_{i,j}=\bar{X}_{j,i}$ for $i>j$. The GUE(N) is defined to be the random matrix $X^N=(X_{i,j})_{1\leqslant i,j\leqslant N}$. It induces the following probability measure on the space $\mathcal{H}_N$ of $N\times N$ Hermitian matrices:

$$P_N(\mathrm{d}H) = Z_N^{-1}\exp\Big(-\frac{1}{2}\operatorname{Tr}\big(H^2\big)\Big)\,\mathrm{d}H \tag{1}$$

where $\mathrm{d}H$ is Lebesgue measure on $\mathcal{H}_N$. In the same way, if $M\geqslant N$ and $A^{N,M}$ is a $N\times M$ matrix whose entries are complex standard independent Gaussian variables, then LUE(N,M) is defined to be the random $N\times N$ matrix $Y^{N,M}=A^{N,M}(A^{N,M})^*$ where $*$ stands for the conjugate of the transposed matrix. Alternatively, LUE(N,M) corresponds to the following measure on $\mathcal{H}_N$:

$$P_{N,M}(\mathrm{d}H) = Z_{N,M}^{-1}(\det H)^{M-N}\exp(-\operatorname{Tr} H)\,\mathbf{1}_{H\geqslant 0}\,\mathrm{d}H. \tag{2}$$

A central-limit theorem which already appeared in the Introduction of [7] asserts that GUE(N) is the limit in distribution of LUE(N,M) as $M\to\infty$ in the following asymptotic regime:

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$$\frac{Y^{N,M} - M \,\mathrm{Id}_N}{\sqrt{M}} \xrightarrow[M\to\infty]{\mathrm{d}} X^N. \tag{3}$$

For connections with this result, see Theorem 2.5 of [2] and a note in Section 5 of [9]. We also state a process-level version of the previous convergence when the Gaussian entries of the matrices are replaced by Brownian motions. The convergence takes place for the trajectories of the eigenvalues.

Next, we make use of this matrix central-limit theorem together with an observation due to Johansson [5] and an invariance principle for a last-passsage time due to Glynn and Whitt [3] in order to recover the following celebrated equality in distribution between the maximal eigenvalue $\lambda_{\max}^N$ of $\mathrm{GUE}(N)$ and some functional of standard N-dimensional Brownian motion $(B_i)_{1\leqslant i\leqslant N}$ as

$$\lambda_{\max}^N \stackrel{\mathrm{d}}{=} \sup_{0=t_0\leqslant\cdots\leqslant t_N=1} \sum_{i=1}^N \big(B_i(t_i) - B_i(t_{i-1})\big). \tag{4}$$

The right-hand side of (4) can be thought of as a last-passage time in an oriented Brownian percolation. Its discrete analogue for an oriented percolation on the sites of $\mathbb{N}^2$ is the object of Johansson's remark. The identity (4) first appeared in [1] and [4]. Very recently, O'Connell and Yor shed a remarkable light on this result in [10]. Their work involves a representation similar to (4) for all the eigenvalues of $\mathrm{GUE}(N)$. We notice here that analogous formulae can be written for all the eigenvalues of $\mathrm{LUE}(N, M)$. On the one hand, seeing the particular expression of these formulae, a central-limit theorem can be established for them and the limit variable Ω is identified in terms of Brownian functionals. On the other hand, the previous formulae for eigenvalues of $\mathrm{LUE}(N, M)$ converge, in the limit given by (3), to the representation found in [10] for $\mathrm{GUE}(N)$ in terms of some path-transformation Γ of Brownian motion. It is not immediately obvious to us that functionals Γ and Ω coincide. In particular, is this identity true pathwise or only in distribution?

The matrix central-limit theorem is presented in Section 2 and its proof is postponed to the last section. In section 3, we described the consequences to eigenvalues representations and the connection with the O'Connell-Yor approach.

2 The central-limit theorem

Here is the basic form of the matrix-central limit theorem:

Theorem 1. *Let $Y^{N,M}$ and X^N be taken respectively from $LUE(N, M)$ and $GUE(N)$. Then*

$$\frac{Y^{N,M} - M \,\mathrm{Id}_N}{\sqrt{M}} \xrightarrow[M\to\infty]{\mathrm{d}} X^N. \tag{5}$$

We turn to the process version of the previous result. Let $A^{N,M} = (A_{i,j})$ be a $N \times M$ matrix whose entries are independent standard complex Brownian motions. The Laguerre process is defined to be $Y^{N,M} = A^{N,M}(A^{N,M})^*$. It is built in exactly the same way as LUE(N,M) but with Brownian motions instead of Gaussian variables. Similarly, we can define the Hermitian Brownian motion X^N as the process extension of GUE(N).

Theorem 2. *If $Y^{N,M}$ is the Laguerre process and $(X^N(t))_{t\geqslant 0}$ is Hermitian Brownian motion, then:*

$$\left(\frac{Y^{N,M}(t) - Mt\,\mathrm{Id}_N}{\sqrt{M}}\right)_{t\geqslant 0} \xrightarrow[M\to\infty]{\mathrm{d}} \left(X^N(t^2)\right)_{t\geqslant 0} \tag{6}$$

in the sense of weak convergence in $\mathcal{C}(\mathbb{R}_+, \mathcal{H}_N)$.

As announced, the proofs of the previous theorems are postponed up to section (4). Theorem 1 is an easy consequence of the usual multi-dimensionnal central-limit theorem. For Theorem 2, our central-limit convergence is shown to follow from a law of large numbers at the level of quadratic variations.

Let us mention the straightforward consequence of Theorems 1 and 2 on the convergence of eigenvalues. If $H \in \mathcal{H}_N$, let us denote by $l_1(H) \leqslant \cdots \leqslant l_N(H)$ its (real) eigenvalues and $l(H) = (l_1(H), \ldots, l_N(H))$. Using the min-max formulas, it is not difficult to see that each l_i is 1-Lipschitz for the Euclidean norm on $\mathcal{H}_N$. Thus, l is continuous on $\mathcal{H}_N$. Therefore, if we set $\mu^{N,M} = l(Y^{N,M})$ and $\lambda^N = l(X^N)$

$$\left(\frac{\mu_i^{N,M} - M}{\sqrt{M}}\right)_{1\leqslant i\leqslant N} \xrightarrow[M\to\infty]{\mathrm{d}} (\lambda_i^N)_{1\leqslant i\leqslant N}\,. \tag{7}$$

With the obvious notations, the process version also takes place:

$$\left(\left(\frac{\mu_i^{N,M}(t) - Mt}{\sqrt{M}}\right)_{1\leqslant i\leqslant N}\right)_{t\geqslant 0} \xrightarrow[M\to\infty]{\mathrm{d}} \left(\left(\lambda_i^N(t^2)\right)_{1\leqslant i\leqslant N}\right)_{t\geqslant 0}. \tag{8}$$

Analogous results hold in the real case of GOE and LOE and they can be proved with the same arguments. To our knowledge, the process version had not been considered in the existing literature.

3 Consequences on representations for eigenvalues

3.1 The largest eigenvalue

Let us first indicate how to recover from (7) the identity

$$\lambda_{\max}^N \stackrel{\mathrm{d}}{=} \sup_{0=t_0\leqslant\cdots\leqslant t_N=1} \sum_{i=1}^{N} \left(B_i(t_i) - B_i(t_{i-1})\right) \tag{9}$$

where $\lambda_{\max}^N = \lambda_N^N$ is the maximal eigenvalue of GUE(N) and $(B_i\,,\,1 \leqslant i \leqslant N)$ is a standard N-dimensional Brownian motion. If $(w_{i,j}\,,\,(i,j) \in (\mathbb{N} \setminus \{0\})^2)$ are i.i.d. exponential variables with parameter one, define

$$H(M,N) = \max\Big\{\sum\nolimits_{(i,j)\in\pi} w_{i,j}\,;\ \pi \in \mathcal{P}(M,N)\Big\} \tag{10}$$

where $\mathcal{P}(M,N)$ is the set of all paths π taking only unit steps in the north-east direction in the rectangle $\{1,\dots,M\} \times \{1,\dots,N\}$. In [5], it is noticed that

$$H(M,N) \stackrel{\mathrm{d}}{=} \mu_{\max}^{M,N} \tag{11}$$

where $\mu_{\max}^{M,N} = \mu_N^{N,M}$ is the largest eigenvalue of LUE(N,M). Now an invariance principle due to Glynn and Whitt in [3] shows that

$$\frac{H(M,N) - M}{\sqrt{M}} \xrightarrow[M\to\infty]{\mathrm{d}} \sup_{0=t_0\leqslant\cdots\leqslant t_N=1} \sum_{i=1}^{N} \big(B_i(t_i) - B_i(t_{i-1})\big). \tag{12}$$

On the other hand, by (7)

$$\frac{\mu_{\max}^{N,M} - M}{\sqrt{M}} \xrightarrow[M\to\infty]{\mathrm{d}} \lambda_{\max}^N. \tag{13}$$

Comparing (11), (12) and (13), we get (9) for free.

In the next section, we will give proofs of more general statements than (11) and (12).

3.2 The other eigenvalues

In fact, Johansson's observation involves all the eigenvalues of LUE(N,M) and not only the largest one. Although it does not appear exactly like that in [5], it takes the following form. First, we need to extend definition (10) as follows: for each k, $1 \leqslant k \leqslant N$, set

$$\begin{aligned} H_k(M,N) = \max\Big\{&\sum\nolimits_{(i,j)\in\pi_1\cup\cdots\cup\pi_k} w_{i,j}\,; \\ &\pi_1,\dots,\pi_k \in \mathcal{P}(M,N),\ \pi_1,\dots,\pi_k \text{ all disjoint}\Big\}. \end{aligned} \tag{14}$$

Then, the link, analogous to (11), with the eigenvalues of LUE(N,M) is expressed by

$$H_k(M,N) \stackrel{\mathrm{d}}{=} \mu_N^{N,M} + \mu_{N-1}^{N,M} + \cdots + \mu_{N-k+1}^{N,M}. \tag{15}$$

In fact, the previous equality in distribution is also valid for the vector $(H_k(M,N))_{1\leqslant k\leqslant N}$ and the corresponding sums of eigenvalues, which gives a representation for all the eigenvalues of LUE(N,M).

Proof of (15). The arguments and notations are taken from Section 2.1 in [5]. Denote by $\mathcal{M}_{M,N}$ the set of $M \times N$ matrices $A = (a_{ij})$ with non-negative integer entries and by $\mathcal{M}^s_{M,N}$ the subset of $A \in \mathcal{M}_{M,N}$ such that $\Sigma(A) = \sum a_{ij} = s$. Let us recall that the Robinson–Schensted–Knuth (RSK) correspondence is a one-to-one mapping from $\mathcal{M}^s_{M,N}$ to the set of pairs (P, Q) of semi-standard Young tableaux of the same shape λ which is a partition of s, where P has elements in $\{1, \dots, N\}$ and Q has elements in $\{1, \dots, M\}$. Since $M \geqslant N$ and since the numbers are strictly increasing down the columns of P, the number of rows of λ is at most N. We will denote by $\mathrm{RSK}(A)$ the pair of Young tableaux associated to a matrix A by the RSK correspondence and by $\lambda(\mathrm{RSK}(A))$ their common shape. The crucial fact about this correspondence is the combinatorial property that, if $\lambda = \lambda(\mathrm{RSK}(A))$, then for all k, $1 \leqslant k \leqslant N$,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = \max\Big\{\sum\nolimits_{(i,j)\in\pi_1\cup\cdots\cup\pi_k} a_{i,j}\,; \quad \pi_1, \dots, \pi_k \in \mathcal{P}(M,N),\ \pi_1, \dots, \pi_k \text{ all disjoint}\Big\}. \tag{16}$$

Now consider a random $M \times N$ matrix X whose entries (x_{ij}) are i.i.d. geometric variables with parameter q. Then for any λ^0 partition of an integer s, we have

$$\mathbb{P}\big\{\lambda\big(\mathrm{RSK}(X)\big) = \lambda^0\big\} = \sum\nolimits_{A\in\mathcal{M}^s_{M,N},\ \lambda(\mathrm{RSK}(A))=\lambda^0} \mathbb{P}\{X = A\}.$$

But for $A \in \mathcal{M}^s_{M,N}$, $\mathbb{P}\{X = A\} = (1-q)^{MN} q^s$ is independent of A, which implies

$$\mathbb{P}\big\{\lambda\big(\mathrm{RSK}(X) = \lambda^0\big)\big\} = (1-q)^{MN} q^{\sum \lambda^0_i}\, L(\lambda^0, M, N)$$

where $L(\lambda^0, M, N) = \sharp\{A \in \mathcal{M}_{M,N},\ \lambda(\mathrm{RSK}(A)) = \lambda^0\}$. Since the RSK mapping is one-to-one

$$L\big(\lambda^0, M, N\big) = Y\big(\lambda^0, M\big)\, Y\big(\lambda^0, N\big)$$

where $Y(\lambda^0, K)$ is just the number of semi-standard Young tableaux of shape λ^0 with elements in $\{1, \dots, K\}$. This cardinal is well-known in combinatorics and finally

$$L\big(\lambda^0, M, N\big) = c_{MN}^{-1} \prod_{1\leqslant i<j\leqslant N} \big(h^0_j - h^0_i\big)^2 \prod_{1\leqslant i\leqslant N} \frac{(h^0_i + M - N)!}{h^0_i!}$$

where $c_{MN} = \prod_{0\leqslant i\leqslant N-1} j!\,(M - N + j)!$ and $h^0_i = \lambda^0_i + N - i$ such that $h_1 > h_2 > \cdots > h_N \geqslant 0$. With the same correspondence as before between h and λ, we can write

$$\begin{aligned}
&\mathbb{P}\big\{h\big(\mathrm{RSK}(X)\big) = h^0\big\} \\
&\qquad = c_{MN}^{-1} \frac{(1-q)^{MN}}{q^{N(N-1)/2}} \prod_{1\leqslant i<j\leqslant N} \big(h^0_j - h^0_i\big)^2 \prod_{1\leqslant i\leqslant N} \frac{(h^0_i + M - N)!}{h^0_i!} \\
&\qquad \overset{\text{def}}{=} \rho_{(M,N,q)}\big(h^0\big).
\end{aligned}$$

Now set $q = 1 - L^{-1}$ and use the notation X_L instead of X to recall the dependence of the distribution on L. An easy asymptotic expansion shows that

$$L^N \rho_{(M,N,1-L^{-1})}(\lfloor Lx \rfloor) \xrightarrow[L\to\infty]{} d_{MN}^{-1} \prod_{1\leqslant i<j\leqslant N} (x_j - x_i)^2 \prod_{1\leqslant i\leqslant N} x_i^{M-N} \mathrm{e}^{-x_i} = \rho_{\mathrm{LUE}(N,M)}(x)$$

where $\rho_{\mathrm{LUE}(N,M)}$ is the joint density of the ordered eigenvalues of $\mathrm{LUE}(N, M)$. This can be used to prove that

$$\frac{1}{L}\, h\big(\mathrm{RSK}(X_L)\big) \xrightarrow[L\to\infty]{\mathrm{d}} \big(\mu_N^{MN}, \mu_{N-1}^{MN}, \ldots, \mu_1^{MN}\big). \tag{17}$$

On the other hand, if x_L is a geometric variable with parameter $1-L^{-1}$, then x_L/L converges in ditribution, when $L \to \infty$, to an exponential variable of parameter one. Therefore, using the link between h and λ together with (16), we have

$$\frac{1}{L}\left(\sum_{i=1}^{k} h_i\big(\mathrm{RSK}(X_L)\big)\right)_{1\leqslant k\leqslant N} \xrightarrow[L\to\infty]{\mathrm{d}} \big(H_k(M,N)\big)_{1\leqslant k\leqslant N}.$$

Comparing with (17), we get the result. □

Now, let us try to adapt what we previously did with $H(M, N)$ and $\mu_{\max}^{M,N}$ to the new quantities $H_k(M, N)$. First, we would like to have an analogue of the Glynn–Whitt invariance principle (12). To avoid cumbersome notations, let us first look at the case $k = 2$, $N = 3$. In this case, the geometry involved in the $H_2(M, 3)$ is simple: we are trying to pick up the largest possible weight by using two north-east disjoint paths in the rectangle $\{1, \ldots, M\} \times \{1, 2, 3\}$. The most favourable configuration corresponds to one path (the bottom one) starting at $(1, 1)$ and first going right. Then it jumps to some point of $\{2, \ldots, M\} \times \{2\}$ and goes horizontally up to $(M, 2)$. The upper path starts at $(1, 2)$, will also jump and go right up to $(M, 3)$. The constraint that our paths must be disjoint forces the x-coordinate of the jump of the bottom path to be larger than that of the jump of the upper path. This corresponds to the obvious figure 1.

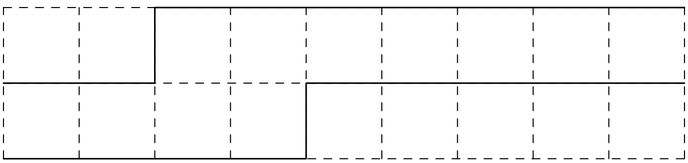

Fig. 1. Configuration of paths in the case $k = 2$ and $N = 3$

This figure suggests that in the Donsker limit of random walks converging to Brownian motion, we will have

$$\begin{aligned}\frac{H_2(M,3)-2M}{\sqrt{M}} &\xrightarrow[M\to\infty]{\mathrm{d}} \Omega_2^{(3)}\\ &\stackrel{\mathrm{def}}{=} \sup_{0\leqslant s\leqslant t\leqslant 1}\big(B_1(t)+B_2(s)+B_2(1)-B_2(t)+B_3(1)-B_3(s)\big)\end{aligned}$$

where (B_1, B_2, B_3) is standard 3-dimensional Brownian motion.

For the case of $k = 2$ and general N, we have the same configuration except that the number of jumps for each path will be $N-2$ so that

$$\begin{aligned}\frac{H_2(M,N)-2M}{\sqrt{M}} &\xrightarrow[M\to\infty]{\mathrm{d}} \Omega_2^{(N)}\\ &\stackrel{\mathrm{def}}{=} \sup\sum_{j=1}^{N}\big(B_j(s_{j-1})-B_j(s_{j-2})+B_j(t_j)-B_j(t_{j-1})\big) \quad (18)\end{aligned}$$

where $(B_j)_{1\leqslant j\leqslant N}$ is a standard N-dimensional Brownian motion and the sup is taken over all subdivisions of $[0,1]$ of the following form:

$$\begin{aligned}0=s_{-1}=s_0=t_0\leqslant s_1\leqslant t_1\leqslant s_2\leqslant t_2\\ \leqslant\cdots\leqslant t_{N-2}\leqslant s_{N-1}=t_{N-1}=s_N=t_N=1.\end{aligned}$$

Proof of limit (18). Let us first consider the case of $H_2(M,N)$:

$$H_2(M,N)=\max\Big\{\sum\nolimits_{(i,j)\in\pi_1\cup\pi_2} w_{i,j},;\ \pi_1,\,\pi_2\in\mathcal{P}(M,N)\,;\ \pi_1,\,\pi_2 \text{ disjoint}\Big\}.$$

Since our paths are disjoint, one (say π_1) is always lower than the other (say π_2): for all $i\in\{1,\cdots,M\}$, $\max\{j\,;\,(i,j)\in\pi_1\}<\min\{j\,;\,(i,j)\in\pi_2\}$. We will denote this by $\pi_1<\pi_2$. Then, it is not difficult to see on a picture that, for any two paths $\pi_1<\pi_2\in\mathcal{P}(M,N)$, one can always find paths $\pi_1'<\pi_2'\in\mathcal{P}(M,N)$ such that $\pi_1\cup\pi_2\subset\pi_1'\cup\pi_2'$, π_1' starts from $(1,1)$, visits $(2,1)$ then finishes in $(M,N-1)$ and π_2' starts from $(2,1)$ and goes up to (M,N). Let us call $\mathcal{P}(M,N)'$ the set of pairs of such paths (π_1',π_2'). Thus

$$H_2(M,N)=\max\Big\{\sum\nolimits_{(i,j)\in\pi_1\cup\pi_2} w_{i,j}\,;\ (\pi_1,\pi_2)\in\mathcal{P}(M,N)'\Big\}.$$

Now two paths $(\pi_1,\pi_2)\in\mathcal{P}(M,N)'$ are uniquely determined by the non-decreasing sequences of their $N-2$ vertical jumps, namely $0\leqslant t_1\leqslant\cdots\leqslant t_{N-2}\leqslant 1$ for π_1 and $0\leqslant s_1\leqslant\cdots\leqslant s_{N-2}\leqslant 1$ for π_2 such that:

- π_1 is horizontal on $[\,\lfloor t_{i-1}M\rfloor,\lfloor t_iM\rfloor\,]\times\{i\}$ and vertical on $\{\lfloor t_iM\rfloor\}\times[i,i{+}1]$,
- π_2 is horizontal on $[\,\lfloor s_{i-1}M\rfloor,\lfloor s_iM\rfloor\,]\times\{i+1\}$ and vertical on $\{\lfloor s_iM\rfloor\}\times[i+1,i+2]$,
- $s_i<t_i$ for all $i\in\{1,\ldots,N-2\}$, this constraint being equivalent to the fact that $\pi_1<\pi_2$.

The weight picked up by two such paths coded by (t_i) and (s_i) is

- $w_{1,1} + w_{2,1} + \cdots + w_{\lfloor t_1 M \rfloor,1}$ on the first floor,
- $w_{1,2} + \cdots + w_{\lfloor s_1 M \rfloor,2} \ + \ w_{\lfloor t_1 M \rfloor,2} + \cdots + w_{\lfloor t_2 M \rfloor,2}$ on the second floor,
- $w_{\lfloor s_1 M \rfloor,3} + \cdots + w_{\lfloor s_2 M \rfloor,3} \ + \ w_{\lfloor t_2 M \rfloor,3} + \cdots + w_{\lfloor t_3 M \rfloor,3}$ on the third floor,
- and so on, up to floor N for which the contribution is $w_{\lfloor s_{N-2} M \rfloor,N} + \cdots + w_{M,N}$.

This yields

$$H_2(M,N) = \sup \sum_{j=1}^{N} \left(\sum_{i=\lfloor s_{j-2} M \rfloor}^{\lfloor s_{j-1} M \rfloor} w_{i,j} \ + \sum_{i=\lfloor t_{j-1} M \rfloor}^{\lfloor t_j M \rfloor} w_{i,j} \right).$$

Hence,

$$\frac{H_2(M,N) - 2M}{\sqrt{M}} = \sup \sum_{j=1}^{N} \left(\frac{\sum_{i=\lfloor s_{j-2} M \rfloor}^{\lfloor s_{j-1} M \rfloor} w_{i,j} - (s_{j-1} - s_{j-2})M}{\sqrt{M}} + \frac{\sum_{i=\lfloor t_{j-1} M \rfloor}^{\lfloor t_j M \rfloor} w_{i,j} - (t_j - t_{j-1})M}{\sqrt{M}} \right).$$

Donsker's principle states that

$$\left(\frac{\sum_{i=1}^{\lfloor sM \rfloor} w_{ij} - sM}{\sqrt{M}} \right)_{1 \leqslant j \leqslant N} \xrightarrow[M \to \infty]{\mathrm{d}} \big(B_j(s)\big)_{1 \leqslant j \leqslant N}$$

where the convergence takes place in the space of cadlag trajectories of the variable $s \in \mathbb{R}_+$ equipped with the Skorohod topology. This allows us to conclude (see [3] for a detailed account on the continuity of our mappings in the Skorohod topology). $\square$

For general k and N, the same pattern works with k disjoint paths having each $N-k$ jumps. This yields the following central-limit behaviour:

$$\frac{H_k(M,N) - kM}{\sqrt{M}} \xrightarrow[M \to \infty]{\mathrm{d}} \Omega_k^{(N)} \stackrel{\mathrm{def}}{=} \sup \sum_{j=1}^{N} \sum_{p=1}^{k} \Big(B_j\big(s_{j-p+1}^p\big) - B_j\big(s_{j-p}^p\big) \Big) \quad (19)$$

where the sup is taken over all subdivisions (s_i^p) of $[0,1]$ of the following form:

$$s_i^p \in [0,1], \ s_i^{p+1} \leqslant s_i^p \leqslant s_{i+1}^p, \ s_i^p = 0 \text{ for } i \leqslant 0 \text{ and } s_i^p = 1 \text{ for } i \geqslant N-k+1.$$

Now, imitating the argument for the $\lambda_{\max}^N$, we obtain that

$$\Omega_k^{(N)} \stackrel{\mathrm{d}}{=} \lambda_N^N + \lambda_{N-1}^N + \cdots + \lambda_{N-k+1}^N \quad (20)$$

where we recall that $\lambda_1^N \leqslant \cdots \leqslant \lambda_N^N$ are the eigenvalues of GUE(N). In fact, the previous equality is also true when considering the vector $(\Omega_k^{(N)})_{1 \leqslant k \leqslant N}$

and the corresponding sums of eigenvalues, which yields a representation for all the eigenvalues of GUE(N).

A representation for the eigenvalues of GUE(N) was already obtained in [10]. Let us compare both representations. Denote by $\mathcal{D}_0(\mathbb{R}_+)$ the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$ and for $f, g \in \mathcal{D}_0(\mathbb{R}_+)$, define $f \otimes g \in \mathcal{D}_0(\mathbb{R}_+)$ and $f \odot g \in \mathcal{D}_0(\mathbb{R}_+)$ by

$$f \otimes g(t) = \inf_{0 \leqslant s \leqslant t} \big(f(s) + g(t) - g(s)\big) \quad \text{and} \quad f \odot g(t) = \sup_{0 \leqslant s \leqslant t} \big(f(s) + g(t) - g(s)\big).$$

By induction on N, define $\Gamma^{(N)} : \mathcal{D}_0(\mathbb{R}_+)^N \to \mathcal{D}_0(\mathbb{R}_+)^N$ by

$$\Gamma^{(2)}(f, g) = (f \otimes g, g \odot f)$$

and for $N > 2$ and $f = (f_1, \ldots, f_N)$

$$\Gamma^{(N)}(f) = \Big(f_1 \otimes \cdots \otimes f_N, \\ \Gamma^{(N-1)}\big(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \ldots, f_N \odot (f_1 \otimes \cdots \otimes f_{N-1})\big)\Big).$$

Then the main result in [10] is:

$$\lambda^N \stackrel{\mathrm{d}}{=} \Gamma^{(N)}(B)(1) \tag{21}$$

where $B = (B_i)_{1 \leqslant i \leqslant N}$ is standard N-dimensional Brownian motion and λ^N is the vector of eigenvalues of GUE(N). In fact, it is proved in [10] that identity (21) is true for the whole processes and not only their marginals at time 1.

Thus

$$\lambda_N^N + \lambda_{N-1}^N + \cdots + \lambda_{N-k+1}^N \stackrel{\mathrm{d}}{=} \Gamma_N^{(N)}(B)(1) + \Gamma_{N-1}^{(N)}(B)(1) + \cdots + \Gamma_{N-k+1}^{(N)}(B)(1).$$

Comparison with (20) gives

$$\Omega_k^{(N)} \stackrel{\mathrm{d}}{=} \Gamma_N^{(N)}(B)(1) + \Gamma_{N-1}^{(N)}(B)(1) + \cdots + \Gamma_{N-k+1}^{(N)}(B)(1). \tag{22}$$

This equality in distribution also holds for the N-vector $(\Omega_k^{(N)})_{1 \leqslant k \leqslant N}$.

Now let us remark that the definition of the components $\Gamma_k^{(N)}$ of $\Gamma^{(N)}$ is quite intricate: it involves a sequence of nested “inf” and “sup”. On the contrary, $\Omega_k^{(N)}$ is only defined by one “sup” but over a complicated sequence of nested subdivisions. We ignore whether these identities are: trivial and uninteresting; already well-known; true for the deterministic formulas (i.e., true when replacing independent Brownian motions by continuous functions) or true only in distribution.

Our concern raises the question about the link between the $\Gamma^{(N)}$ introduced in [10] and the Robinson–Schensted–Knuth correspondence that gave birth to our $\Omega^{(N)}$. Very interesting results in this direction are obtained by O’Connell in [11].

Finally, let us notice that the heart of our arguments to get the previous representations is the identity (14). The proof presented here is taken from [5] and is organized in two steps: first the computation of the joint density for $(H_k(M,N))_{1\leqslant k\leqslant N}$ by combinatorial means and second the observation that this density coincides with the eigenvalue density of $\mathrm{LUE}(N,M)$. It would be tempting to get a deeper understanding of this result. This would all amount to obtaining a representation for non-colliding squared Bessel processes.

4 Proofs

Proof of Theorem 1. Let us denote by $Z^{(j)}$ the matrix $(A_{k,j}A_{l,j})_{1\leqslant k,j\leqslant N}$ so that $Y^{N,M}=\sum_{j=1}^{N}Z^{(j)}$. Since $(Z^{(j)})_{j\geqslant 1}$ are independent $\mathbb{L}^2$ random variables with commun law Z, the multi-dimensionnal central-limit theorem states that:

$$\frac{1}{\sqrt{M}}\left(\sum_{j=1}^{M}Z^{(j)}-M\,\mathrm{Id}_N\right)\xrightarrow[M\to\infty]{\mathrm{d}}\mathcal{N}_{\mathcal{H}_N}(0,\mathbb{C}\mathrm{ov}\,Z).$$

Thus, we just need to check that the covariance structure of Z coincides with that of X taken from $\mathrm{GUE}(N)$. In this case, $\mathbb{C}\mathrm{ov}(X_{a,b},X_{c,d})=\delta_{a,d}\delta_{c,b}$ for $1\leqslant a,b,c,d\leqslant N$. For Z, $\mathbb{C}\mathrm{ov}(Z_{a,b},Z_{c,d})=\mathbb{E}(A_{a,1}\bar{A}_{b,1}A_{c,1}\bar{A}_{d,1})-\delta_{a,b}\delta_{c,d}$. We have to distinguish three cases to compute $e=\mathbb{E}(A_{a,1}\bar{A}_{b,1}A_{c,1}\bar{A}_{d,1})$: either all indexes are equal ($a=b=c=d$) and $e=\mathbb{E}(|A_{a,1}|^4)=2$, or else one index is different from the three others and $e=0$, or else they are equal by pairs, which gives rise to three more situations: $a=b\neq c=d$ for which $e=\mathbb{E}(|A_{a,1}|^2)\mathbb{E}(|A_{c,1}|^2)=1$, $a=c\neq b=d$ for which $e=\mathbb{E}(A_{a,1}^2)\mathbb{E}(A_{b,1}^2)=0$ and $a=d\neq b=c$ for which $e=\mathbb{E}(A_{a,1}^2)\mathbb{E}(A_{b,1}^2)=1$. In each case, $e-\delta_{a,b}\delta_{c,d}=\delta_{a,d}\delta_{c,b}$ which is our result. □

Remark 1. In fact, one can also give an elementary proof by direct computation on the density of $Y^{N,M}$ just using Stirling's formula and the following asymptotic expansion $\log\det(\mathrm{Id}_N+\varepsilon H)=\varepsilon\,\mathrm{Tr}\,H-(\varepsilon^2/2)\,\mathrm{Tr}\,H^2+\mathcal{O}(\varepsilon^3)$ for small ε.

Proof of Theorem 2. We will write A instead of $A^{M,N}$. For $1\leqslant i\leqslant N$, $1\leqslant j\leqslant M$, the superscript ij when applied to a matrix stands for its entry at line i and column j. The value at time t of any process x will be denoted either $x(t)$ or x_t. Let us set

$$Z_M(t)=\frac{Y^{N,M}(t)-Mt\,\mathrm{Id}_N}{\sqrt{M}}=\frac{AA^*(t)-Mt\,\mathrm{Id}_N}{\sqrt{M}}\,.$$

Then

$$Z_M^{ij}=\frac{1}{\sqrt{M}}\sum_{k=1}^{M}\bigl(A^{ik}\bar{A}^{jk}-Mt\delta_{ij}\bigr),\quad \mathrm{d}Z_M^{ij}=\frac{1}{\sqrt{M}}\sum_{k=1}^{M}\bigl(A^{ik}\,\mathrm{d}\bar{A}^{jk}+\bar{A}^{jk}\,\mathrm{d}A^{ik}\bigr),$$

which implies

$$\mathrm{d}Z_M^{ij} \cdot \mathrm{d}Z_M^{i'j'} = \frac{1}{M} \sum_{k=1}^{M} \big(A^{ik} \bar{A}^{j'k} \delta_{i'j} + \bar{A}^{jk} A^{i'k} \delta_{ij'} \big)\, \mathrm{d}t.$$

The quadratic variation follows to be:

$$\big\langle Z_M^{ij}, Z_M^{i'j'} \big\rangle_t = \frac{1}{M} \sum_{k=1}^{M} \int_0^t \big(A_s^{ik} \bar{A}_s^{j'k} \delta_{i'j} + \bar{A}_s^{jk} A_s^{i'k} \delta_{ij'} \big)\, \mathrm{d}s.$$

By the classical law of large numbers, we get that this converges almost surely to:

$$\int_0^t \Big(\mathbb{E}\big(A_s^{i1} \bar{A}_s^{j'1}\big) \delta_{i'j} + \mathbb{E}\big(\bar{A}_s^{j1} A_s^{i'1}\big) \delta_{ij'} \Big)\, \mathrm{d}s = \int_0^t \delta_{ij'} \delta_{i'j} 2s \,\mathrm{d}s = t^2 \delta_{ij'} \delta_{i'j}.$$

Note that the previous formula shows that, in the limit, the quadratic variation is 0 if $i \neq j'$ and $i' \neq j$, which is obvious even for finite M without calculations. However, if for instance $i = j'$ and $i' \neq j$, then the quadratic variation is not 0 for finite M and only becomes null in the limit. This is some form of asymptotic independence.

First, let us prove tightness of the process Z_M on any fixed finite interval of time $[0, T]$. It is sufficient to prove tightness for every component, let us do so for Z_M^{11} for example (Z_M^{11} is real). We will apply Aldous' criterion (see [8]). Since $Z_M^{11}(0) = 0$ for all M, it is enough to check that, for all $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{M \to \infty} \sup_{\tau,\, 0 \leqslant \theta \leqslant \delta} \mathbb{P}\big\{ \big| Z_M^{11}(\tau + \theta) - Z_M^{11}(\tau) \big| \geqslant \varepsilon \big\} = 0 \tag{23}$$

where the sup is taken over all stopping times τ bounded by T. For τ such a stopping time, $\varepsilon > 0$ and $0 \leqslant \theta \leqslant \delta \leqslant 1$, we have

$$\begin{aligned}
\mathbb{P}\big\{ \big| Z_M^{11}(\tau + \theta) - Z_M^{11}(\tau) \big| \geqslant \varepsilon \big\} &\leqslant \frac{1}{\varepsilon^2}\, \mathbb{E}\Big(\big(Z_M^{11}(\tau + \theta) - Z_M^{11}(\tau) \big)^2 \Big) \\
&= \frac{1}{\varepsilon^2}\, \mathbb{E}\Big(\int_\tau^{\tau+\theta} \mathrm{d}\big\langle Z_M^{11}, Z_M^{11} \big\rangle_t \Big) = \frac{2}{M\varepsilon^2} \sum_{k=1}^{M} \mathbb{E}\Big(\int_\tau^{\tau+\theta} |A_s^{1k}|^2 \,\mathrm{d}s \Big) \\
&\leqslant \frac{2}{M\varepsilon^2} \sum_{k=1}^{M} \mathbb{E}\Big(\theta \sup_{0 \leqslant s \leqslant T+1} \big| A_s^{1k} \big|^2 \Big) = \frac{2\theta}{\varepsilon^2}\, \mathbb{E}\Big(\sup_{0 \leqslant s \leqslant T+1} \big| A_s^{11} \big|^2 \Big).
\end{aligned}$$

Since $c_T = \mathbb{E}(\sup_{0 \leqslant s \leqslant T+1} |A_s^{11}|^2) < \infty$, then

$$\limsup_{M \to \infty} \sup_{\tau,\, 0 \leqslant \theta \leqslant \delta} \mathbb{P}\big\{ \big| Z_M^{11}(\tau + \theta) - Z_M^{11}(\tau) \big| \geqslant \varepsilon \big\} \leqslant \frac{2\delta\, c_T}{\varepsilon^2}\,.$$

This last line obviously proves (23).

Let us now see that the finite-dimensionnal distributions converge to the appropriate limit. Let us first fix i, j and look at the component $Z_M^{ij} = (x_M + \sqrt{-1}\, y_M)/\sqrt{2}$. We can write

$$\langle x_M, y_M\rangle_t = 0, \qquad \langle x_M, x_M\rangle_t = \langle y_M, y_M\rangle_t = \frac{1}{M}\sum_{k=1}^{M}\int_0^t \alpha_s^k \,\mathrm{d}s \tag{24}$$

where $\alpha_s^k = |A_s^{ik}|^2 + |A_s^{jk}|^2$. We are going to consider x_M. Let us fix $T \geqslant 0$. For any $(\nu_1, \dots, \nu_n) \in [-T, T]^n$ and any $0 = t_0 < t_1 < \cdots < t_n \leqslant T$, we have to prove that

$$\mathbb{E}\biggl(\exp\Bigl(i\sum_{j=1}^{n}\nu_j\bigl(x_M(t_j) - x_M(t_{j-1})\bigr)\Bigr)\biggr) \xrightarrow[M\to\infty]{} \exp\Bigl(\sum_{j=1}^{n}\frac{\nu_j^2}{2}\bigl(t_j^2 - t_{j-1}^2\bigr)\Bigr). \tag{25}$$

We can always suppose $|t_j - t_{j-1}| \leqslant \delta$ where δ will be chosen later and will only depend on T (and not on n). We will prove property (25) by induction on n. For $n = 0$, there is nothing to prove. Suppose it is true for $n-1$. Denote by $(\mathcal{F}_t)_{t\geqslant 0}$ the filtration associated to the process A. Then write:

$$\begin{aligned}\mathbb{E}&\Bigl(\mathrm{e}^{i\sum_{j=1}^{n}\nu_j(x_M(t_j)-x_M(t_{j-1}))}\Bigr)\\ &= \mathbb{E}\Bigl(\mathrm{e}^{i\sum_{j=1}^{n-1}\nu_j(x_M(t_j)-x_M(t_{j-1}))}\,\mathbb{E}\bigl(\mathrm{e}^{i\nu_n(x_M(t_n)-x_M(t_{n-1}))} \bigm| \mathcal{F}_{t_{n-1}}\bigr)\Bigr).\end{aligned} \tag{26}$$

We define the martingale $\mathcal{M}_t = \mathrm{e}^{i\nu_n x_M(t) - \frac{\nu_n^2}{2}\langle x_M, x_M\rangle_t}$. Hence

$$\mathbb{E}\bigl(\mathrm{e}^{i\nu_n(x_M(t_n)-x_M(t_{n-1}))} \bigm| \mathcal{F}_{t_{n-1}}\bigr) = \mathbb{E}\Bigl(\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}}\,\mathrm{e}^{\frac{\nu_n^2}{2}\langle x_M, x_M\rangle_{t_{n-1}}^{t_n}} \Bigm| \mathcal{F}_{t_{n-1}}\Bigr)$$

with the notation $\langle x, x\rangle_s^t = \langle x, x\rangle_t - \langle x, x\rangle_s$. This yields

$$\begin{aligned}\mathrm{e}^{-\frac{\nu_n^2}{2}(t_n^2-t_{n-1}^2)}\,&\mathbb{E}\bigl(\mathrm{e}^{i\nu_n(x_M(t_n)-x_M(t_{n-1}))} \bigm| \mathcal{F}_{t_{n-1}}\bigr) - 1\\ &= \mathbb{E}\Bigl(\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}}\,\zeta_M \Bigm| \mathcal{F}_{t_{n-1}}\Bigr)\end{aligned} \tag{27}$$

where we set $\zeta_M = \mathrm{e}^{\frac{\nu_n^2}{2}(\langle x_M, x_M\rangle_{t_{n-1}}^{t_n} - (t_n^2 - t_{n-1}^2))} - 1$. Using that $|\mathrm{e}^z - 1| \leqslant |z|\mathrm{e}^{|z|}$, we deduce that

$$|\zeta_M| \leqslant K\,\bigl|\langle x_M, x_M\rangle_{t_{n-1}}^{t_n} - \bigl(t_n^2 - t_{n-1}^2\bigr)\bigr|\,\mathrm{e}^{\frac{\nu_n^2}{2}\langle x_M, x_M\rangle_{t_{n-1}}^{t_n}}$$

where $K = \nu_n^2/2$. The Cauchy–Schwarz inequality implies that

$$\mathbb{E}(|\zeta_M|) \leqslant K \left(\mathbb{E}\Big(\langle x_M, x_M\rangle_{t_{n-1}}^{t_n} - \big(t_n^2 - t_{n-1}^2\big)\Big)^2\right)^{1/2} \left(\mathbb{E}\Big(\mathrm{e}^{\nu_n^2 \langle x_M, x_M\rangle_{t_{n-1}}^{t_n}}\Big)\right)^{1/2}.$$

By convexity of the function $x \to \mathrm{e}^x$:

$$\mathrm{e}^{\nu_n^2 \langle x_M, x_M\rangle_{t_{n-1}}^{t_n}} = \exp\Big(\frac{1}{M}\sum_{k=1}^{M} \nu_n^2 \int_{t_{n-1}}^{t_n} \alpha_u^k \,\mathrm{d}u\Big) \leqslant \frac{1}{M}\sum_{k=1}^{M} \mathrm{e}^{\nu_n^2 (t_n - t_{n-1}) \sup\limits_{0\leqslant u\leqslant t_n} \alpha_u^k}$$

and thus

$$\begin{aligned}\mathbb{E}&\Big(\mathrm{e}^{\nu_n^2 \langle x_M, x_M\rangle_{t_{n-1}}^{t_n}}\Big)\\ &\leqslant \frac{1}{M}\sum_{k=1}^{M} \mathbb{E}\Big(\mathrm{e}^{\nu_n^2 (t_n - t_{n-1}) \sup\limits_{0\leqslant u\leqslant t_n} \alpha_u^k}\Big) = \mathbb{E}\Big(\mathrm{e}^{\nu_n^2 (t_n - t_{n-1}) \sup\limits_{0\leqslant u\leqslant t_n} \alpha_u^1}\Big).\end{aligned}$$

Now let us recall that $\alpha_u^1 = |A_u^{i1}|^2 + |A_u^{j1}|^2$, which means that α^1 has the same law as a sum of squares of four independent Brownian motions. It is then easy to see that there exists $\delta > 0$ (depending only on T) such that

$$\mathbb{E}\Big(\exp\Big(T^2\delta \sup_{0\leqslant u\leqslant T} \alpha_u^1\Big)\Big) < \infty.$$

With this choice of δ,

$$K' = \mathbb{E}\Big(\mathrm{e}^{\nu_n^2 (t_n - t_{n-1}) \sup\limits_{0\leqslant u\leqslant t_n} \alpha_u^1}\Big) < \infty$$

and thus:

$$\mathbb{E}(|\zeta_M|) \leqslant K\,K' \left(\mathbb{E}\Big(\langle x_M, x_M\rangle_{t_{n-1}}^{t_n} - \big(t_n^2 - t_{n-1}^2\big)\Big)^2\right)^{1/2} \xrightarrow[M\to\infty]{} 0$$

(by the law of large numbers for square-integrable independent variables). Since $|\mathcal{M}_{t_n}/\mathcal{M}_{t_{n-1}}| \leqslant 1$, we also have

$$\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}}\,\zeta_M \xrightarrow[M\to\infty]{\mathbb{L}^1} 0.$$

Therefore

$$\mathbb{E}\Big(\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}}\,\zeta_M \;\Big|\; \mathcal{F}_{t_{n-1}}\Big) \xrightarrow[M\to\infty]{\mathbb{L}^1} 0. \tag{28}$$

In turn, by looking at (27), this means that

$$\mathbb{E}\big(\mathrm{e}^{i\nu_n (x_M(t_n) - x_M(t_{n-1}))} \;\big|\; \mathcal{F}_{t_{n-1}}\big) \xrightarrow[M\to\infty]{\mathbb{L}^1} \mathrm{e}^{\frac{\nu_n^2}{2}(t_n^2 - t_{n-1}^2)}.$$

Now, plug this convergence and the induction hypothesis for $n-1$ into (26) to get the result for n.

The same is true for y_M. To check that the finite-dimensionnal distributions of Z_M^{ij} have the right convergence, we would have to prove that:

$$\mathbb{E}\left(\exp\left(i\sum_{i=1}^{n}\nu_i\big(x_M(t_i)-x_M(t_{i-1})\big)+\mu_i\big(y_M(t_i)-y_M(t_{i-1})\big)\right)\right) \xrightarrow[M\to\infty]{} \exp\left(\sum_{i=1}^{n}\frac{\nu_i^2+\mu_i^2}{2}\,(t_i^2-t_{i-1}^2)\right). \tag{29}$$

But since $\langle x_M, y_M\rangle = 0$,

$$\mathcal{M}_t=\exp\left(i\big(\nu_n x_M(t)+\mu_n y_M(t)\big)-\frac{\nu_n^2}{2}\langle x_M,x_M\rangle_t-\frac{\mu_n^2}{2}\langle y_M,y_M\rangle_t\right)$$

is a martingale and the reasoning is exactly the same as the previous one.

Finally, let us look at the asymptotic independence. For the sake of simplicity, let us take only two entries. Set for example $x_M = Z_M^{11}$ and $y_M = \sqrt{2}\,\mathrm{Re}(Z_M^{12})$. Then we have to prove (29) for our new x_M, y_M. Since $\langle x_M, y_M\rangle \neq 0$, $\mathcal{M}_t$ previously defined is no more a martingale. But

$$\mathcal{N}_t=\exp\left(i\big(\nu_n x_M(t)+\mu_n y_M(t)\big) -\frac{\nu_n^2}{2}\langle x_M,x_M\rangle_t-\frac{\mu_n^2}{2}\langle y_M,y_M\rangle_t-\nu_n\mu_n\langle x_M,y_M\rangle_t\right)$$

is a martingale and the fact that $\langle x_M, y_M\rangle_t \xrightarrow[M\to\infty]{\mathbb{L}^2} 0$ allows us to go along the same lines as before. □

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