Brownian Sheet Local Time and Bubbles

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Summary. We establish a law of large numbers relating the number of "bubbles" contained in a bounded time domain and local time on that domain. The result is analagous to the behaviour of Brownian motion.

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1 Introduction

The Brownian sheet is a centred Gaussian Process indexed by $\mathbf{s} = (s_1, s_2) \in \mathbb{R}_+^2$. Its covariance is given by

$$\operatorname{cov}\big(W(\mathbf{s}), W(\mathbf{t})\big) = (s_1 \wedge t_1)(s_2 \wedge t_2);$$

this and path continuity fully define the process.

This note concerns *bubbles*; these are components of $\{\mathbf{t} : W(\mathbf{t}) \neq 0\}$. We think of them as natural higher dimensional time analogues of excursions away from 0. We shall refer to an *x-bubble* for $x > 0$ as a bubble on which the maximum value taken by the Brownian sheet lies in the interval $(x, 2x)$. For the restricted purposes of this article we will also require that bubbles be components that are entirely contained in the rectangle $[0,1]^2$. In referring to components whose maximal value is in the interval $(x, 2x)$ but which are not necessarily contained in $[0,1]^2$ we use the term *x-component*.

The local time (at zero) for rectangle $[x_1, x_2] \times [y_1, y_2]$ is given by

$$L([x_1, x_2] \times [y_1, y_2]) = \lim_{t \downarrow 0} \frac{1}{2\varepsilon} \int_{x_1}^{x_2} \int_{y_1}^{y_2} I_{|W(s,t)| < \varepsilon} \, \mathrm{d}s \, \mathrm{d}t$$

where I denotes and will denote the indicator function. Of course it has to be proven that this limit exists (see e.g. [E], [R]). In fact these works show that

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the function L is a continuous function for the four arguments x_1, x_2, y_1 and y_2 and that local time yields a measure $L(\mathrm{d}s, \mathrm{d}t) = \mathrm{d}L(s,t)$ supported on the 0-level set. Sometimes we will regard time rectangles as intervals in two space and write $[\mathbf{x}, \mathbf{y}]$ instead of $[x_1, x_2] \times [y_1, y_2]$. Always it will be understood that x_1 the first component of $\mathbf{x}$ is less than or equal to y_1 the first co-ordinate of $\mathbf{y}$ and similarly for the second co-ordinates. The results here are proved directly for the local time for $[0,1]^2$, $L = L([0,1]^2)$ and for components contained in this rectangle, but it will be clear that the arguments and techniques extend to arbitrary bounded time rectangles.

The Lebesgue measure of the time that the sheet spends at zero is zero. Nonetheless we think of L as measuring the amount of time spent around zero. For Brownian motion there is a clear and beautiful theory relating local time to excursions (see e.g. [RY], chapters VI and XII, or [RW], chapter VI, especially pages 414–424) and it is well known, for instance, that as $x \downarrow 0$ the number of excursions by time t of Brownian motion from 0 having maximum in $(x, 2x)$, $(N_x(t))$, satisfies

$$xN_x(t) - \frac{1}{4}L(t) \xrightarrow{\text{pr}} 0, \tag{1}$$

as x tends to zero. Here $L(t)$ denotes the local time of the Brownian motion over the interval $[0,t]$ defined in an analogous manner to the above.

We wish to prove:

Theorem 1. *Let N_x be the number of x-bubbles of a Brownian sheet, and let the local time of the Brownian sheet on $[0,1]^2$ be L. Then*

$$x^3 N_x - c \int_{[0,1]^2} st \,\mathrm{d}L(s,t) \xrightarrow{\text{pr}} 0, \tag{2}$$

as x tends to zero, for some strictly positive constant c.

This result and Corollary 2 (proven in Section 1) with scaling easily lead to

Corollary 1. *Let M_x be the number of bubbles of a Brownian sheet which have maximum value greater than x and let the local time of the Brownian sheet on $[0,1]^2$ be L. Then*

$$x^3 M_x - c' \int_{[0,1]^2} st \,\mathrm{d}L(s,t) \xrightarrow{\text{pr}} 0,$$

as x tends to zero, for some strictly positive constant c'.

Corollary 1 was conjectured in [Kh]. In that paper it was shown that this conjecture is of the right order in the sense that for all $\varepsilon > 0$, as $x \to 0$,

$$x^{3+\varepsilon} M_x \xrightarrow{\text{pr}} 0 \qquad \text{and} \qquad x^{3-\varepsilon} M_x \xrightarrow{\text{pr}} \infty.$$

Many of the ideas used in this note originate in that paper.

In the remainder of the introduction we will give our guiding heuristic as to why this should be true and then discuss the overall approach to the problem that we will follow.

One way to see (1) or at least to see why N_x, the number of excursions having maximum in $(x, 2x)$ contained in interval $(0, 1)$, should be of order $1/x$ is to note that when Brownian motion hits, say, x then it has chance $1/2$ of returning to zero before reaching $(2x, \infty)$. One would expect then that a reasonable proportion of time spent by Brownian motion in $[0, x]$ would correspond to time for excursions to $[x, 2x]$. But the existence of a continuous local time at 0 for Brownian motion means that the time spent in $[0, x]$ up to time 1 is $xL(1) + o(x)$. Furthermore the length of an excursion having maximum value in $[x, 2x]$ is of order x^2. Dividing $xL(1)$ by x^2 almost gives us (1). Given the linear time this heuristic can easily be turned into a rigourous proof of (1). For (2) the heuristic is similar with the expected size of a bubble having maximum size in $[x, 2x]$ now being $(x^2)^2 = x^4$ instead of x^2. But it is not so straight forward to construct a proof.

Our approach is first to remove some troublesome extreme cases: we show that there are "not too many" x-bubbles of size $\leqslant \gamma x^4$ for γ small, since this would require many large deviations for the Brownian sheet, and that there are "not too many" x-bubbles of large diameter, meaning of diameter $\geqslant Mx^2$ for M large.

This reduction means that the bubbles that count are for the most part regular-sized components. Locally (for small x) the Brownian sheet in square $[t_1, t_1 + Kx^2] \times [t_2, t_2 + Kx^2]$ is (after due rescaling of time) like the difference of two independent Brownian motion's process $X(s, t) = B(s) - B'(t)$ where the Brownian motions are of speeds t_2 and t_1 respectively. The reason that our result concerns $\int_{[0,1]^2} st \, \mathrm{d}L(s, t)$ and not L comes from this time inhomogeneity of the process.

The difference of two Brownian motions process has a nice bubble theory discussed in [DW], [DW2] and this enables us to compare as $x \to 0$, the distribution of the number of x-bubbles of area in $[\gamma x^4, \infty]$ entirely contained in $[t_1, t_1 + Kx^2] \times [t_2, t_2 + Kx^2]$ conditional on $W(t_1, t_2) = yx$ to the distribution of the number of 1-bubbles for X contained in $[0, 1]^2$ of area at least γ, given $X(0, 0) = y$.

As an additional hygene measure we also show that the number of x-bubbles near the co-ordinate axes is small. The basic argument then is to divide up $[\varepsilon, 1]^2$ into a grid of rectangles which are small (though their dimensions will not depend on x). On these small rectangles the Brownian sheet will be almost time homogenous. We will then divide up a given rectangle R into a grid of horizontal spacing length $c_1 x^2$ and of vertical length $c_2 x^2$. We will argue that if the (i,j) grid-rectangle has bottom left vertex $\mathbf{t}_{jk}$, then for some function g, the number of bubbles in R is approximately equal to $\sum_{jk} g(W(\mathbf{t}_{jk}/x))$ for some bounded function g of compact support. We then employ the following simple result,

Lemma 1. *Let g be a bounded function of compact support. For a rectangle R bounded away from the axes if R is divided up into a grid $\{\mathbf{t}_{jk}\}$ of horizontal spacing c_1x^2 and vertical c_2x^2, then*

$$x^3 \sum_{jk} g\big(W(\mathbf{t}_{jk})/x\big) \quad \textit{converges in probability to} \quad \frac{1}{c_1c_2}\left(\int_{-\infty}^{\infty} g(u)du\right)L(R)$$

as x tends to zero.

Proof. We fix an interval $[-K, K]$ and consider the bounded Borel functions with this interval as their support. We identify this vector space with the Borel measureable functions on $[-K, K]$. Let H be the collection of bounded Borel functions g on $[-K, K]$ so that

$$x^3 \sum_{jk} g\big(W(\mathbf{t}_{jk})/x\big) \quad \text{converges in probability to} \quad \frac{1}{c_1c_2}\left(\int_{-\infty}^{\infty} g(u)\,\mathrm{d}u\right)L(R)$$

as x tends to zero. By linearity of the sums and the integral we immediately see that H is a vector space. Furthermore, as is easily seen by a second moment argument, the result holds true for functions of the form $g(u) = I_{c_1 \leqslant u \leqslant c_2}$, $-K \leqslant c_1 < c_2 \leqslant K$ (where $I_{(\,.\,)}$ denotes an indicator function). This includes constant functions. Also if g_n is an increasing sequence of functions n H converging pointwise to bounded function g, then by bounded convergence

$$\int g_n(u)\,\mathrm{d}u \longrightarrow \int g(u)\,\mathrm{d}u.$$

Equally the expectation of $x^3 \sum_{jk} g(W(\mathbf{t}_{jk})/x) - x^3 \sum_{jk} g_n(W(\mathbf{t}_{jk})/x)$ converges to 0 as $n \to \infty$. We conclude that under these conditions g must also be in H and so as a direct consequence of the function version of the Monotone Class Theorem (see e.g. [RY], Theorem 0.2.2) one concludes that for every Borel function g supported on $[-K, K]$ the lemma holds. The entire lemma follows by the arbitrariness of K. □

The paper is planned as follows: Section 2 is devoted to establishing that one may discard from consideration bubbles that are of too small an area, of too large a diameter or that are too close to the time axes. The third section considers the conditionally expected number of x-bubbles, reasonable in the above sense, that occur in a time rectangle $[t_1, t_1 + Kx^2/t_2, t_2, t_2 + Kx^2/t_1]$ given that $W(\mathbf{t}) = yx$. Finally the elements are gathered together to finish the proof of Theorem 1 in the final section.

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2 Section Two

Loosely speaking, we wish to show that "most" x-bubbles contain a square of time of side length of the order x^2. We also wish to show that "most" bubbles of size x are of diameter of the order x^2 and not close to the time axes.

Proposition 1 below bounds the number of x-bubbles having small area. Essentially it follows because a bubble with a small area requires extreme variation in the sheet.

Corollary 2 below bounds the number of x-bubbles close to the axes, while Corollary 3 deals with bubbles of large diameter.

Proposition 1. *For all $\delta > 0$, there exists $\varepsilon > 0$ so that the expected number of x-components that attain value x within $[0,1]^2$ and do not contain a square of side $x^2\varepsilon$ with a vertex within $[0,1]^2$ is bounded by δ/x^3.*

Remark 1. The methods used in the proof of this proposition apply equally well if the Brownian sheet is replaced by the difference of two Brownian motions. The proof of this proposition thus implies that for the difference of two Brownian motions and a nonrandom x the number of x-components on any bounded rectangle is a.s. finite.

This proposition is proven by showing that corresponding to each bad or small bubble is an "extreme" excursion for a Brownian motion

$$B^{s,u}(t) = W(s,t), \quad t \geqslant 0, \qquad \text{or} \qquad B^{t,h}(s) = W(s,t), \quad s \geqslant 0.$$

Note in this proposition we are considering bubbles that are not necessarily strictly contained in $[0,1]^2$.

We easily obtain the following corollary.

Corollary 2. *For all $\delta > 0$, there exists $\varepsilon > 0$ so that the expected number of x-components that attain value x within $[0,\varepsilon] \times [0,1] \cup [0,1] \times [0,\varepsilon]$ is less than δ/x^3.*

In turn Corollary 2 begets

Corollary 3. *For all $\delta > 0$, there exists $M < \infty$ so that the expected number of x-components attaining value x within $[0,1]^2$ and of diameter $> Mx^2$ is bounded by δ/x^3.*

We now set ourselves to showing some technical results with the ultimate goal of proving the above results.

Definition 1. *An excursion $e = (e_1, e_2)$ of maximum in $(x/2, 2x]$ for a Brownian motion B, is in $A(x,n)$ if*

$$\sup_{e_1 \leqslant p < q \leqslant e_2,\, |p-q| < x^2 2^{-n}} |B(p) - B(q)| > \frac{x}{32}\,.$$

Lemma 2. *For a Brownian motion of speed $\sigma \leqslant 1$, there exists a strictly positive constant k (uniformly over speed $\sigma \leqslant 1$) so that the probability that an excursion whose maximum is in $[x/2, 2x]$ (randomly chosen according to excursion measure) is in $A(x,n)$ is bounded by $\mathrm{e}^{-2^n k}$.*

Proof. By scaling it suffices to treat the case σ equal to 1 and so for the proof B will be a speed one Brownian motion. Before directly considering a Brownian excursion we deal with Brownian motion on time interval $[0, x^2 2^n]$. We choose this time interval which is very long for an excursion since the probability that a Brownian excursion whose maximal value lies in $(x/2, 2x)$ should have lifetime greater than $(2^n - 1)x^2$ is certainly less than $K' \mathrm{e}^{-c' 2^n}$ for finite strictly positive constants K', c'.

Let $T = \inf\{t : |B(t) - B(s)| > x/32 \text{ for some } s \in [t - x^2 2^{-n}, t]\}$.

Let $S = \inf\{t > T : t \in x^2 2^{-n}\mathbb{Z}\}$ be the first time after T that is in the lattice $x^2 2^{-n}\mathbb{Z}$.

By the strong Markov property and symmetry, we have

$$P\left\{ \sup_{S-2\cdot x^2 2^{-n} \leqslant s \leqslant S} |B_s - B_S| > \frac{x}{32} \,\middle|\, \mathcal{F}_T \right\} \geqslant 1/2.$$

Thus for a Brownian motion B

$$\begin{aligned}
&P\left\{ \sup\nolimits_{0<|p-q|<x^2 2^{-n} 0<p,q<x^2 2^{+n}} |B_p - B_q| > \frac{x}{32} \right\} = P\{T < x^2 2^n\} \\
&\qquad \leqslant 2\, P\left\{ \bigcup\nolimits_{t \in x^2 2^{-n}\mathbb{Z} \cap [0, x^2 2^n + x^2)} V_t \right\} \\
&\qquad\quad \text{(where } V_g \text{ is the event } \{\sup\nolimits_{g - 2x^2 2^{-n} \leqslant s \leqslant g} |B_s - B_g| > x/32\}) \\
&\qquad \leqslant 3 \times 4^n\, P\{V_{2x^2 2^{-n}}\} \\
&\qquad \leqslant 6 \times 4^n\, P\left\{ N(0,1) > \frac{\sqrt{2^n}}{32\sqrt{2}} \right\} \\
&\qquad \leqslant K 4^n \mathrm{e}^{-2^n/4096} \leqslant K \mathrm{e}^{-c 2^n}.
\end{aligned}$$

Given this we note that with probability bounded away from zero, B will begin an excursion from 0, having maximum value in $[x/2, 2x]$, in time $[0, x^2]$ and that, outside of probability $K\mathrm{e}^{-c' 2^n}$ this excursion will be completed by time $x^2 2^n$. Thus the claimed result is shown since the inequality need only be proven for large n. □

Corollary 4. *For $s \in [0,1]$, let $Z(s,n,h)$ be the number of excursions in $A(x,n)$ by Brownian motion $B^{s,h}$ that originate in $[0,1]$ (similarly for $Z(s,n,u)$). Then*

$$E\left[\sum_{s \in x^2 2^{-n}\mathbb{Z} \cap (0,1]} Z(s,n,h) + Z(s,n,u) \right] \leqslant \frac{K\, \mathrm{e}^{-c 2^n}}{x^3}.$$

Proof. By symmetry we need only consider the expectation of

$$\sum_{s \in x^2 2^{-n}\mathbb{Z} \cap (0,1]} Z(s,n,h).$$

Fix $s \in x^2 2^{-n}\mathbb{Z} \cap (0,1]$. By Maisonneuve's formula for Brownian excursions from zero and the bound of Lemma 2,

$$E[Z(s,n,h)] \leqslant \mathrm{e}^{-k2^n}\,\frac{1}{2x}\,E\big[L_1^{h,s}\big] \leqslant \mathrm{e}^{-k2^n}\,\frac{1}{x}\sqrt{\frac{1}{2\pi s}},$$

where $E[L_1^{h,s}]$ is the local time at zero for Brownian motion $B^{s,h}$. Thus

$$E\Bigg[\sum_{s\in x^2 2^{-n}\mathbb{Z}\cap(0,1]} Z(s,n,h)\Bigg] \leqslant \mathrm{e}^{-k2^n}\,\frac{K}{x^3}\Bigg(x^2\sum_{s\in x^2 2^{-n}\mathbb{Z}\cap(0,1]}\sqrt{\frac{1}{s}}\Bigg).$$

From which the result is immediate. □

Recall $B^{s,u} = W(s,\,.\,)$, $B^{t,h} = W(\,.\,,t)$. For, say, $B^{s,u}$ with s a fixed positive, we want to consider how many excursions $e=(e_1,e_2)$ of $B^{s,u}$ to a maximum value in $(x/2,2x)$ are such that

$$\sup_{\substack{0<p-q<2^{-n}x^2\ p,\\ q\in e}}\ \sup_{\substack{0<|s-s'|<2^{-n}x^2,\\ s<3/2}} |W(s,p)-W(s',p)-W(s,q)+W(s',q)| \geqslant \frac{x}{32}\,. \tag{3}$$

Let this number be $X^s(x,n)$. Let the analagous quantity for $B^{s,h}$ be $Y^s(x,n)$.

Lemma 3. *For s fixed and positive, $0\leqslant x\leqslant 1$ and $X^s(x,n)$ as above, $E[X^s(x,n)] = E[Y^s(x,n)] \leqslant \kappa\,\mathrm{e}^{-c2^n}$ for finite and positive κ, c not depending on $x\leqslant 1$.*

Proof. Let us define, with respect to filtration

$$\mathcal{F}_r = \{W(t_1,t_2) : 0\leqslant t_1<\infty,\ 0\leqslant t_2\leqslant r\},$$

stopping times $(T_i)_{i\geqslant 0}$ by $T_0=0$ and

$$\begin{aligned} T_i = \inf\{v > T_{i-1} : \exists\, T_{i-1}\leqslant u < v \text{ such that, for some } |s-s'|<2^{-n}x^2,\\ |W(s,u)-W(s,v)-W(s',n)+W(s',v)| \geqslant x/32\}.\end{aligned}$$

Then

$$X^s(x,n) \leqslant \sup\{j : T_j \leqslant 2\}.$$

But $P\{T_1\leqslant y\}$ (by the Orey–Pruitt maximal inequality, [OP]) is dominated by

$$\begin{aligned} &8\Big(\frac{y}{2^{-n}x^2}+1\Big)P\big\{\big|N\big(0,2^{-2n}x^4\big)\big| > x/32\big\}\\ &\qquad\leqslant K\Big(\frac{y}{2^{-n}x^2}+1\Big)\mathrm{e}^{-\frac{c}{x^2}2^{2n}} \leqslant K\mathrm{e}^{-c2^n} \qquad \text{for } y\leqslant 2.\end{aligned}$$

Since $T_i - T_{i-1}$ are i.i.d. random variables the result follows from standard arguments on geometric random variables. □

The basic idea underlying the proof of Proposition 1 is that for every x-component G in which W attains value x inside $[0,1]^2$ (even though G itself may not be entirely contained in $[0,1]^2$), there is by definition $\mathbf{t} = (t_1, t_2) \in G \cap [0,1]^2$ so that $W(\mathbf{t}) = x$. If $|G|$, the area of G, is small then it must be the case that for

$$v_1 = \inf\{t > t_1 : |W(t, t_2) - x| > x/8\}$$
$$v_2 = \inf\{t > t_2 : |W(t_1, t) - x| > x/8\},$$

either:

(i) $\min_i (v_i - t_i)$ is small;

(ii) we have a large white noise contribution for some rectangle from bottom-left vertex equal to $\mathbf{t}$.

This will enable us, for some s and some positive integer n to associate to G an excursion to $x/2$ for $B^{s_L,h}$ or $B^{s_L,u}$ for which there is extreme behaviour for $s_L \in x^2\mathbb{Z}2^{-n}$ covered by Lemma 3 or Corollary 4.

We will now make this specific.

Proof of Proposition 1. Suppose a x-component as above, G, has area less or equal to $x^2 2^{-2N_0}$, where 2^{-N_0} will be the ε in the statement of the proposition and will be large but not depending on x. Choose in an arbitrary manner $\mathbf{t}$ in G at which the value of the sheet equals x. For $\mathbf{t}, v_1, v_2$ as above either

$$\min_i v_i - t_i \leqslant x^2 2^{-N_0} \tag{4}$$

or

$$\begin{aligned} v_i - t_i \geqslant x^2 2^{-N_0} \qquad &\text{for } i = 1, 2, \qquad \text{but} \\ W(\mathbf{s}) = 0 \qquad &\text{for some } \mathbf{s} \in [t_1, t_1 + x^2 2^{-N_0}] \times [t_2, t_2 + x^2 2^{-N_0}]. \end{aligned} \tag{5}$$

To prove the proposition it will suffice to bound the expectation of the number of G, $\mathbf{t}$ for which (5) is true and to bound the expectation of the number of G, $\mathbf{t}$ for which (4) holds.

We first treat case (4). We split it up into

$$\min(v_i - t_i) \in (x^2 2^{-(n+1)}, x^2 2^{-n}] \qquad \text{for } n \geqslant N_0.$$

We suppose without loss of generality that

$$v_1 - t_1 = \min(v_i - t_i) \in (x^2 2^{-(n+1)}, x^2 2^{-n}].$$

Let $\mathbf{s} = (s_1, s_2)$ be the "smallest" element of $\mathbb{Z}^2 x^2 2^{-(n+1)}$ in the square $[t_1, v_1] \times [t_2, t_2 + v_1 - t_1]$. We claim that for Brownian motion $B^{s_2,h}$ there must be some kind of large deviation associated with the excursion of $B^{s_2,h}$ containing t_1 (which excursion necessarily corresponds to a line segment contained in G). Necessarily the time point $(t_1, s_2) \in G$ and $|W(t_1, s_2) - x| \leqslant x/8$. If for some $s \in [t_1, v_1]$, $|W(t_1, s_2) - W(s, s_2)| > x/32$, then the excursion of $B^{s_2,h}$ containing time point t_1 is in $A(x,n)$. Suppose on the contrary that for all $s \in [t_1, v_1]$, $|W(t_1,s_2) - W(s,s_2)| \leqslant x/32$ and in particular $|W(t_1,s_2) - W(v_1,s_2)| \leqslant x/32$.

In this case we have $|W(t_1,s_2) - W(v_1,s_2) - W(t_1,t_2) + W(v_1,t_2)| \geqslant x/8 - x/32 > x/32$ and the excursion of $B^{s_2,h}$ containing t_1 makes a contribution to $Y^{s_2}(x,n)$ or to $X^{s_2}(x,n)$. By Lemma 3 and Corollary 4 we have that the number of such excursions is bounded for any fixed $s \in (x^2/2^n)\mathbb{Z}$ by

$$2\,\frac{1}{x^2}\,2^n\left(\frac{\kappa\,\mathrm{e}^{-c2^n}}{x} + \kappa'\,\mathrm{e}^{-c'2^n}\right),$$

summing over $n \geqslant N_0$ we obtain a bound $\leqslant \delta/(10x^3)$ if N_0 has been fixed sufficiently large (independently of x).

Now consider (5). As before we let $\mathbf{s} = (s_1, s_2)$ be the smallest element of $2^{-(N_0+1)}x^2\mathbb{Z}$ in $[t_1, t_1 + 2^{-N_0}x^2] \times [t_2, t_2 + 2^{-N_0}x^2]$.

Now, for all $t \in [t_2, t_2 + 2^{-N_0}x^2]$, we have (by the definition of v_2) that $W(t_1,t) \geqslant 7x/8$ and $\leqslant 9x/8$. In particular $W(t_1,s_2) = B^{s_L,h}(t_1) \in [7x/8, 9x/8]$. Thus, provided that for e the excursion of $B^{s_2,h}$ to x containing time t_1 it is the case that,

$$\sup_{0\leqslant p\leqslant q\leqslant 2^{-N_0}x^2\ p,\,q\in e} \left|B^{s_2,h}(p) - B^{s_2,h}(q)\right| < x/8,$$

we have

$$\frac{6x}{8} \leqslant B^{s_2,h}(s) \leqslant 10x/8 \qquad \text{for all } s \in \left[t_1, t_1 + x^2 2^{-N_0}\right].$$

So, if for some $\mathbf{u} = (u_1, u_2) \in [t_1, t_1 + x^2 2^{-N_0}] \times [t_2, t_2 + x^2 2^{-N_0}]$ we have $W(\mathbf{u}) = 0$, then

$$\begin{aligned} &W(u_1,u_2) - W(u_1,s_2) - W(t_1,u_2) + W(t_1,s_2) \\ &\qquad\qquad \leqslant 0 - 6x/8 - 7x/8 + 9x/8 = -x/2. \end{aligned}$$

Thus again we have for excursion e of B^{s_2} containing t_1, that

$$\sup_{\substack{p,q\in e \\ |p-q|\leqslant x^2 2^{-N_0}}} \ \sup_{|t-s_2|\leqslant x^2 2^{-N_0}} |W(p,t) - W(p,s_2) - W(q,t) + W(q,s_2)| \geqslant x/2.$$

The expected number of such excursions again by Lemma 3 and Corollary 4 (and hence the expected number of bubbles of size $\leqslant 2^{-2N_0}/x^3$) is bounded by

$$\frac{\kappa 2^{4N_0}}{x^2}\left(\frac{\mathrm{e}^{-\kappa 2^{N_0}}}{x} + \mathrm{e}^{-\kappa 2^{N_0}}\right) < \frac{\delta}{10x^3}$$

if N_0 has been fixed sufficiently large. $\square$

Proof of Corollary 2.

$$\begin{aligned}
&E[\#\text{ of } x\text{-bubbles}] \\
&\qquad \leqslant \frac{\delta}{x^3} + E\big[\#\text{ of } x\text{-bubbles containing a square of side } 2^{-N_0}x^2\big] \\
&\qquad \leqslant \frac{\delta}{x^3} + E\big[\big|\{t \in [0,2]^2 : |W| \in [0,2x]\}\big|\big]/2^{-2N_0}x^4 \\
&\qquad \leqslant \frac{\kappa}{x^3}.
\end{aligned}$$

By the scaling properties of centred gaussian variables the process defined on $(s,t) \in [0,1]^2$

$$Y(s,t) = \varepsilon^{-1/2}W(\varepsilon s, t)$$

is equal in law to W. Thus the expectation of the number of $x/\varepsilon^{1/2}$-bubbles attaining value $x/\varepsilon^{1/2}$ in $[0,1]^2$ for process Y = expectation of the number of x-bubbles of W attaining value x in $[0,\varepsilon] \times [0,1]$. But the former quantity is bounded by

$$\frac{\kappa}{(x/\varepsilon^{1/2})^3} = \frac{\kappa\varepsilon^3}{x^3} \leqslant \frac{\delta}{10x^3}$$

for ε small. Thus by symmetry the expected number of x-bubbles attaining value x in $[0,\varepsilon] \times [0,1] \cup [0,1] \times [0,\varepsilon]$ is bounded by $\delta/(5x^3)$ if ε is small. □

Proof of Corollary 3. Given $\delta > 0$, choose ε so small that the expected number of x-bubbles attaining value x in $[0,\varepsilon] \times [0,1] \cup [0,1] \times [0,\varepsilon]$ is bounded by $\delta/(x^3 3)$. Also choose N_0 sufficiently large that the expected number of x-bubbles attaining value x in $[0,1]^2$ that do not contain a square of side $x^2 2^{-N_0}$ with bottom-left vertex in $[0,1]^2$ where $W = x$ is bounded by $\delta/(x^3 3)$.

Now consider a bubble G which is not in the above two collections. By definition G contains a square of side length $x^2 2^{-N_0}$ and centre within $[\varepsilon,1]^2$. Within this square is (at least) one point of $x^2 2^{-(N_0+1)}\mathbb{Z}$. Thus every such bubble G of diameter at least Mx^2 contains a point $\mathbf{s}$ in $x^2 2^{-(N_0+1)}\mathbb{Z}$, so that:

(i) $0 < W(\mathbf{s}) < 2x$;

(ii) $\mathbf{s}$ is not surrounded by a negative W circuit in

$$\left[s_1 - \frac{Mx^2}{3}, s_1 + \frac{Mx^2}{3}\right] \times \left[s_2 - \frac{Mx^2}{3}, s_2 + \frac{Mx^2}{3}\right].$$

But by [K], proof of Theorem 1.1, page 269, the expected number of such points is bounded by

$$2(x^2 2^{-N_0+1})^{-2}\,\frac{2x}{\varepsilon}\,F(M\varepsilon)$$

where $F(y) \to 0$ as $y \to \infty$. The result follows by taking M so large that $F(M\varepsilon) < \varepsilon\delta/(2^{2N_0}3)$. □

3 Section Three

In this section we wish to establish a weak convergence result on a bounded functional of continuous functions from an interval $[0, K]^2$ to the real line. While this functional will not be continuous everywhere we will show that it is a.s. continuous at sites chosen according to the difference of independant Brownian motions process. The importance of this is that locally the Brownian sheet resembles this process. This part of the paper relies heavily on the Dalang–Walsh algorithm introduced in [DW2]. In this section and only this section *bubble* will denote a component without spatial restriction.

We consider the expected number of x-bubbles of area greater or equal to $\gamma x^4/(t_1 t_2)$ contained in

$$\left[t_1,\, t_1 + \frac{Kx^2}{t_2}\right] \times \left[t_2,\, t_2 + \frac{Kx^2}{t_1}\right]$$

given that $W(t_1, t_2) = cx$, for x tending to zero. K will be large but fixed, c fixed, $\mathbf{t} \in [\varepsilon, 1]^2$, $\varepsilon > 0$. Every thing in this section is a simple derivation from ideas and results of [DW2].

We wish to show that as x tends to zero this tends to the expected number of (size at least γ) 1-bubbles in $[0, K]^2$ for a process $X(s, t) = B(s) - B'(t)$ where B, B' are standard independent Brownian motions conditioned on $X(0, 0) = c$.

There are various problems to address. Firstly, while it is trivial that,

$$V(s_1, s_2) = \frac{1}{x} W\Big(\Big[t_1 + x^2\, \frac{s_1}{t_2}\,,\; t_2 + x^2\, \frac{s_2}{t_1}\Big]\Big)\Big|_{\mathbf{s}\in[0,K]^2}$$

conditional on $W(t_1, t_2) = cx$ tends to $X(\mathbf{s})|_{\mathbf{s}\in[0,K]^2}$ conditional on $X(0, 0) = c$, the two dimensional data functional $F(\omega)$ equal to the number of 1-bubbles of area greater or equal to γ contained in $[0, K]^2$ for ω need not be continuous. It might be that as $w_n \to w$ uniformly in $[0, K]^2$, in the limit a single w_n bubbles splits into 2 distinct (necessarily touching) w bubbles. Equally it could be that w_n bubbles of area strictly less than γ converge to a w bubble of area equal to γ. It could be that w_n bubbles which are not contained in $[0, K]^2$ "converge" to a ω-bubble entirely contained in $[0, K]^2$ or that $w_n(1 - \varepsilon_n)$ bubbles which are not w_n 1-bubbles "converge" to a 1-bubble for path w.

The latter difficulties could be dealt with via a "smoothing" of our path functional but the first is difficult: we do not know whether distinct positive bubbles may touch for the Brownian sheet. Nevertheless we shall see that the path functional for w chosen according to a law of X (conditional on $X(0, 0)$) is a.s. continuous at w.

In considering a component G of process X (now considered to be indexed by $(-\infty, \infty)^2$), [DW2] note that if the (a.s.) unique maximum of G occurs at $\mathbf{t} = (t_1, t_2)$ then if we consider $X(s, t_2) = B(s) - B'(t_2)$ for s in a neighborhood of t_1, we see that B must assume a local maximum at t_1. We define (s_1^1, s_2^1) to be the largest interval on which $X(s_1,t_2)>0$. Necessarily $B(s_1^1)=B(s_2^1)=B'(t_2)$ and (s_1^1,s_2^1) is an excursion of B above value $B'(t_2)$.

Similarly we have that B' has a local minimum at t_2 and we have $t_2\in[s_1^2,s_2^2]$, an excursion of B' below value $B(t_1)$. Let us call $=\{t_1\}\times[s_1^2,s_2^2]\cup[s_1^1,s_2^1]\times\{t_2\}$ *the cross* of G and let $[s_1^1,s_2^1]\times[s_1^2,s_2^2]$ be *the rectangle generated* by G.

Lemma 4. *A.s. every positive bubble G is such that a.s. for all $\beta>0$ there is a circuit surrounding the rectangle R generated by G which is within $\beta>0$ of R and on which $X<0$.*

Proof. Let the (countable) values of the local minima of B' be $y_1,y_2,\dots,y_i,\dots$ With probability 1, for all i the excursions e of B above y_i have the property that if $e=(e_1,e_2)$ then for all $\beta>0$ there exist $t_1\in(e_1-\beta,e_1)$ and $t_2\in(e_2,e_2+\beta)$ so that $B(t_1)<y_i$, $B(t_2)<y_i$.

Similarly for excursions e$'$ of B' below local maxima $x_1,x_2,\dots$ of B, we have if e$'=(e_1',e_2')$ then a.s. for all $\beta>0$, there exist $t_1'\in(e_1'-\beta,e_1')$, $t_2'\in(e_2',e_2'+\beta)$ so that $B'(t_1')>x_i$, $B'(t_2')>x_i$.

Thus considering $X=B-B'$ for a cross $C=[s_1^1,s_2^1]\times[s_1^2,s_2^2]$ centred at $\mathbf{t}=(t_1,t_2)$, we have for β small there exists $g_1^i\in(s_1^i-\beta,s_1^i)$, $g_2^i\in(s_2^i,s_2^i+\beta)$ so that $B(g_1^1)<B'(t_2)$, $B(g_2^1)<B'(t_2)$, $B'(g_1^2)>B(t_1)$, $B'(g_2^2)>B(t_1)$. Now $B(t_1)$ is the maximum value of B on $[s_1^1,s_2^1]$ and if β is small we will have $B(s)\leqslant B(t_1)$ on $[s_1^1-\beta,s_2^1+\beta]$ and so $X(s,t)=B(s)-B'(t)$ will be strictly negative on

$$[s_1^1-\beta,s_2^1+\beta]\times\{g_1^2\}\qquad\text{and}\qquad[s_1^1-\beta,s_2^1+\beta]\times\{g_2^2\}.$$

Similarly X will be strictly negative on

$$\{g_1^1\}\times[s_1^2-\beta,s_2^2+\beta]\qquad\text{and}\qquad\{g_2^1\}\times[s_1^2-\beta,s_2^2+\beta]$$

so we can take as our circuit

$$\big(\{g_1^1\}\times[g_1^2,g_2^2]\big)\cup\big(\{g_2^1\}\times[g_1^2,g_2^2]\big)\cup\big([g_1^1,g_2^1]\times\{g_1^2\}\big)\cup\big([g_1^1,g_2^1]\times\{g_2^2\}\big).\ \square$$

Lemma 5. *For X restricted to a square S, any two distinct x bubbles are a.s. non touching.*

Proof. For simplicity we take the square to be $[0,1]^2$ and we consider two distinct bubbles contained in this square. In general the restriction to $[0,1]^2$ means that the crosses may intersect $\partial[0,1]^2$, the boundary of $[0,1]^2$. But still the proof of Lemma 4 applies to the parts of the cross contained in $[0,1]^2$. Let the two x-bubbles be G_1, G_2. Let the crosses corresponding to G_i be C_i. First C_1 cannot cross C_2 as this would mean that G_1 and G_2 are the same component, nor can C_1 and C_2 touch as a moments thought rules out. If the rectangles R_i generated by G_i are disjoint then Lemma 4 yields a circuit separating R_1, R_2 on which $X<0$ and so we must have that G_1, G_2 are a strictly positive distance apart. So we suppose neither of these. Now the intervals (s_1^{11}, s_2^{11}) and (s_1^{12}, s_2^{12}) are excursions above a certain level. Thus if these intervals overlap, it must be the case that one interval contains the other. Similarly for (s_1^{21}, s_2^{21}) and (s_1^{22}, s_2^{22}). A moments thought convinces that we must have either the intervals defining R_1 contained in those defining R_2 or vice-versa, with strict inclusion (we are considering the case R_i do not intersect $\partial[0,1]^2$). We suppose without loss of generality

$$\left(s_1^{11}, s_2^{11}\right) \supset \left(s_1^{12}, s_2^{12}\right) \qquad \text{and} \qquad \left(s_1^{21}, s_2^{21}\right) \supset \left(s_1^{22}, s_2^{22}\right).$$

In this case there is a circuit D surrounding C_2 and disjoint from C_1 on which $X < 0$ by Lemma 4. Thus $G_2 \subset D$ is contained in the interior of D and hence is a strictly positive distance from G_1. The cases where $s_i^{11} = 0$ or 1 are dealt with similarly. Thus in considering x-bubbles on a square $[0,1]^2$ for process X, we have a.s. (see the Remark after the statement of Proposition 1) that there are only a finite number of x-bubbles $G_1, \ldots, G_R$ and associated with each G_i is an exterior circuit C^i and interior circuits C^j, $j \in I(i)1$ so that $X < 0$ on C^i, C^j, $j \in I(i)$ and if $X(\mathbf{t}) > x$ for $\mathbf{t}$ inside C^i and outside C^j, $j \in I(i)$, then $\mathbf{t} \in G_i$. □

We wish to show:

Lemma 6. *If $\omega : [\mathbf{0}, \mathbf{m}] \to \mathbb{R}$ is chosen according to the law of X, then for a.e. ω if $G_1, G_2, \ldots, G_R$ are the x-bubbles of ω of area at least γx^4, and if $w_n \to \omega$ in uniform norm, then, for all i, $|G_i| \neq \gamma x^4$ (here $|\,.\,|$ denotes area) and for n large we have w_n has precisely R x-bubbles $G_1^n, \ldots, G_R^n$ of area at least γx^4 so that:*

(i) *for all i, $G_i \subset (\mathbf{0}, \mathbf{m}) \iff G_i^n \subset (\mathbf{0}, \mathbf{m})$;*

(ii) *for all i, $|G_i^n| \to |G_i|$.*

Proof. It is easy to see that a.s. no x-bubble has area exactly γx^4, we leave this to the reader. Let the (a.s. finite) x-bubbles of ω be $G_1, G_2, \ldots, G_r$, $r \geqslant R$. We can and will assume that $\mathbf{m}$ is equal to (1,1) and also the following:

(i) $\bar{G}_i \cap \partial[0,1]^2 \neq \phi \Rightarrow \omega(\mathbf{t}) > 0$ for some $\mathbf{t} \in \bar{G}_i \cap \partial[0,1]^2$;

(ii) G_i satisfy the circuit property above;

(iii) there exists $\sigma > 0$ so that ω has no bubbles having maximum value in $[x - \sigma, x + \sigma]$;

(iv) $|\{\mathbf{t} : w(\mathbf{t}) = 0\}| = 0$.

Obviously by compactness for n large $w_n < 0$ on each circuit C^i. Also for each i if we choose $\mathbf{t}_i \in G_i$ with $w(\mathbf{t}_i) \geqslant x + \sigma$ (σ as in (iii) above), then $w_n(\mathbf{t}_i) \geqslant x + \sigma/2$ for n large. Define (for n large) G_i^n to be the x-bubble of w_n containing $\mathbf{t}_i$. By the observation for circuits C^i, we have that for n large these r-bubbles are distinct. We first establish that for n large there does not exist a further distinct x-bubble G_{r+1}^n. Suppose not. Taking a subsequence if required, we can assume that there exist $\mathbf{t}_{r+1}^n$ for each n large so that $\mathbf{t}_{r+1}^n \not\subset \bigcup_{i=1}^r G_i^n$ and $w_n(\mathbf{t}_{r+1}^n) \geqslant x$.

If $\mathbf{t}_{r+1}$ is a limit point of the $\mathbf{t}_{r+1}^n$, then $w(\mathbf{t}_{r+1}) \geqslant x$. And so by (iii) $\mathbf{t}_{r+1}$ belongs to bubble G_i for some i. So there exists a path γ_i from $\mathbf{t}_i$ to $\mathbf{t}_{r+1}$ on which $w > 0$ implies that for n sufficiently large we have (by uniform convergence):

(i) $w_n(\mathbf{s}) > 0$ on γ_i;

(ii) $w_n(\mathbf{s}) > 0$ on a neighbourhood of $\mathbf{t}_{r+1}$.

This implies that for infinitely many n we have $\mathbf{t}_{r+1}^n \in G_i^n$. This contradiction implies that for n large there are only r distinct bubbles for ω_n.

By a similar argument it is clear that

$$\limsup |G_i^n| \geqslant |G_i|; \qquad \text{it remains to show:} \qquad \limsup |G_i^n| \leqslant |G_i|.$$

We assume not. Taking a subsequence if necessary we assume

$$\forall n,\ |G_i^n| > |G_i| + c, \qquad \text{for some } c > 0.$$

By property (iv) and uniform convergence we have that there exists $h > 0$ so that for large n

$$\left|\{\mathbf{t} : |w_n(\mathbf{t})| < 3h\}\right| < c/3$$

and

$$\left|\hat{h}G_i\right| < |G_i| + c/3.$$

where $\hat{h}G_i = \{\mathbf{t} : d(t, G_i) \leqslant h\}$.

So we can find for all large n, $t_1^n \in G_i^n$. So that

$$w^n(t_i^n) \geqslant h \qquad \text{and} \qquad d(t_i^n, G_i) \geqslant h.$$

Let t_i^∞ be a limit point of the t_i^n. Then $w(t_i^\infty) \geqslant h > 0$, $t_i^\infty \notin G_i$. But therefore t_i^∞ is in a x-bubble for path w of size at least $h/2$ distinct from G_i this yields a contradiction in the usual way. □

From this and Prohorov's theorem (see e.g. [EK]), we deduce:

Theorem 2. *If X^n is a process on rectangle $[\mathbf{0}, \mathbf{m}] \subseteq [0, M]^2$ so that*

$$X^n(0,0) = x_n \longrightarrow x$$
$$X^n(s,t) = B_1^n(s) + B_2^n(t) + V(s,t) + X^n(0,0)$$

where

$$B_1^n,\ B_2^n \text{ are independent} \quad \text{and} \quad B_1^n, B_2^n \xrightarrow{\mathrm{D}} \text{Brownian motions } B_1, B_2$$

and

$$\sup_{s,t} |V(s,t)| \xrightarrow{\mathrm{pr}} 0,$$

then the distribution of the number of 1*-bubbles of* X^n *on* $[0, \mathbf{m}]$ *which have size at least* γ *and are contained in* $(0, \mathbf{m})$ *converge to the distribution of the corresponding number for the difference of Brownian motions process* $X(s,t) = B_1(s) - B_2(t) + x$.

Define $g^\gamma(x, \mathbf{m})$ to be this number we record some elementary results and bounds for g.

Lemma 7. (1) $g^\gamma(c, \mathbf{m})$ *is continuous in* m *for* c, γ *fixed.*

(2) *For* c *such that* $(|c| - 2)^2 > m_1 + m_2$,

$$g^\gamma(c, \mathbf{m}) \leqslant \frac{m_1 m_2}{\sqrt{2}\pi\gamma} \exp\left(\frac{-(|c|-2)^2}{2(m_1+m_2)}\right).$$

We relate Theorem 2.1 to the Brownian sheet.

Lemma 8. *Fix* $\varepsilon > 0$. *Let (as* $n \to \infty$*)* $\mathbf{t}^n \to \mathbf{t} \in [\varepsilon, 1]^2$, $x_n \to 0$, $m_1^n, m_2^n \to M$, $c_n \to c$ *then the conditional expectation of the number of* x_n*-bubbles contained in*

$$(\mathbf{t}^n, \mathbf{t}^n + x^2(m_1^n/t_2^n, m_2^n/t_1^n))$$

of size $\geqslant \gamma x_n^4/(t_1 t_2)$ *conditional on* $W(\mathbf{t}^n) = c_n x_n$ *converges to* $g^\gamma(c, \mathbf{m})$ *as* n *tends to infinity.*

We also have:

Lemma 9. *For all* $\mathbf{t} \in [\varepsilon, 1]^2$, M, γ *fixed and* x *small we have that the conditional expectation of the number of* x*-bubbles contained in*

$$\big(\mathbf{t}, \mathbf{t} + (M/t_2, M/t_1)x^2\big)$$

of size $\geqslant \gamma/(t_1 t_2) x^4$ *conditional on* $W(\mathbf{t})/x = K$ *is bounded by*

$$(cM^2/\gamma)\exp\big(-(K-2)^2/(5M)\big)$$

for some c *not depending on* K, M.

4 Section Four

We wish to prove Theorem 1. It will be sufficient to show that for δ fixed but arbitrarily small, we can write N_x as $N_x' + N_x''$ where for strictly positive constant $c(\delta)$, $N_x' x^3 - c(\delta)L \to 0$ in probability and where N_x'' is a positive random quantity of expectation bounded by $C\delta/x^3$ where C depends neither on x nor on δ.

Let us fix $0 < \delta << 1$. Now fix $\varepsilon > 0$ so that the expected number of x-bubbles which attain value x within $[0, \varepsilon] \times [0, 1] \cup [0, 1] \times [0, \varepsilon]$ is less than $\delta/(10^{10} x^3)$. By Corollary 2 such ε exists. Let m' be such that the expected number of x-bubbles attaining value x within $[0,1]^2$ and of diameter $> m'x^2$ is bounded by $\delta/(10^{10}x^3)$. Such m' exists by Corollary 3. Let N_0 be so large that the expected number of x-bubbles that attain value x within $[0,1]^2$ and do not contain a square of side $x^2 2^{-N_0}$ with a vertex within $[0,1]^2$ is bounded by $\delta/(10^{10}x^3)$. Let $\gamma = \delta^2\varepsilon^2 2^{-2N_0}/10^{10}$. Now choose m so large that

$$\frac{m'}{m} < \frac{\delta^3}{10^{10}} 2^{-2N_0}\varepsilon^3.$$

Fix K so that for c the constant of Lemma 9, $(K-2)^2/5m > K/4$, $\sum_{r\geqslant 0} c 2^r \mathrm{e}^{-2^r K/4} \leqslant \delta\varepsilon 2^{-2N_0}/(m^2 10^{10})$.

Divide up $[\varepsilon,1]^2$ into a finite number of rectangles R^i, $i \in I$, with the property that, for all i, $R^i = [\mathbf{s}^i, \mathbf{t}^i]$ satisfies

$$\frac{(\mathbf{t}^i)_1}{(\mathbf{s}^i)_1} \quad \frac{(\mathbf{t}^i)_2}{(\mathbf{s}^i)_2} \leqslant 1+\gamma.$$

We wish to show that as $x \to 0$, the number of x-bubbles that intersect boundary δR_i has small expectation and that $N_x(R_i)$ the number of x-bubbles which are contained inside R_i is close to (up to terms of order $\delta L(R_i)$)

$$\int_{R_i} st \,\mathrm{d}L(st)$$

in probability.

To economize on notation, we drop the i suffix and consider a rectangle R contained inside $[\varepsilon,1]^2$.

Given x small, we divide up $R = [x_1,x_2]\times[y_1,y_2]$ into equal rectangles of horizontal side equal to

$$\inf\left\{r : r > \frac{mx^2}{y_1} \text{ such that } \frac{(x_2-x_1)}{r} \in \mathbb{Z}\right\}$$

and similarly of vertical side

$$\inf\left\{r : r > \frac{mx^2}{x_1} \text{ such that } \frac{(y_2-y_1)}{r} \in \mathbb{Z}\right\}$$

Let the grid points be (t_i, s_j) $i = 1,\dots,N$, $j = 1,\dots,M$, with

$$t_{i+1} - t_i > 0 \text{ and constant in } i, \qquad s_{j+1} - s_j > 0 \text{ and constant in } j.$$

Let $\Delta_{i,j}$, $(i,j) \in [1,N]\times[1,M]$, be the rectangle from the grid with left-bottom vertex (t_i,s_j). Let rectangle $\Delta'_{i,j} \subseteq \Delta_{i,j}$ have bottom left vertex equal to (t_i,s_j) and have horizontal side length $(s_1/s_j)(t_2-t_1)$ and vertical side length $(t_1/t_i)(s_2-s_1)$.

Let X_{ij} be the number of x-bubbles contained in $\Delta'_{i,j}$, of size at least $\gamma x^4/t_i s_j$. By way of motivation for the introduction of the subrectangles $\Delta'_{i,j}$, note that Theorem 2 may be applied to the conditional law of the X_{ij} as x tends to zero so that the conditional laws have the same distribution. Let X_{ij}^K be the number of x-bubbles contained in $\Delta'_{i,j}$ if $|W(t_i, s_j)| < Kx$, and equal to 0 otherwise.

Lemma 10. *For K as fixed above and all x sufficiently small*

$$E\Big[\sum_{ij}(X_{ij} - X_{ij}^K)\Big] \leqslant \frac{\delta}{10^9 x^3}\,|R|$$

Proof. Let Z_r, $r = 0, 1, 2, \ldots$, be equal to

$$\sum_{i,j} X_{i,j} I_{|W(t_i,s_j)| \in [2^r Kx, 2^{r+1}Kx)}\ ;$$

then, for x sufficiently small,

$$\begin{aligned} E[Z_r] &= \sum_{i,j} E\big[X_{ij} I_{|W(t_i,s_j)| \in [2^r Kx, 2^{r+1}Kx)}\big] \\ &= \sum_{i,j} P\big\{|W(t_i, s_j)| \in [2^r Kx, 2^{r+1}Kx)\big\} \\ &\qquad \times E\big[X_{ij} \bigm| |W(t_i, s_j)| \in [2^r Kx, 2^{r+1}Kx)\big] \\ &\leqslant 2^{2N_0} m^2 \sum_{i,j} \frac{K2^r x}{\varepsilon}\, \mathrm{e}^{-(K2^r - 2)^2/(5m)} \\ &\leqslant |R|\, c\, 2^r 2^{2N_0}\, \mathrm{e}^{-K2^r/4}/(\varepsilon x^3). \end{aligned}$$

Where for the penultimate inequality we used Lemma 9 and our choice of K. Thus $E[\sum_{r=0}^\infty Z_r] \leqslant \delta|R|/(10^{10}x^3)$ by our choice of K. □

We have introduced a collection of squares Δ_{ij} with side length of order mx^2. We will shortly consider $\sum_{ij} X_{ij}^K$, which after Lemma 10 is close to $\sum_{ij} X_{ij}$. We have to treat the remaining bubbles which achieve value $x/2$ within R. A priori, this number could be extremely large, in principle of order $|R|/x^3$. However, given Corollaries 2 and 3, we need only consider bubbles of diameter bounded by $m'x^2$ having area at least γx^4. Given this we are dealing with bubbles close to the edges of the grid, which is to say bubbles entirely contained in a non-random set of very small Lebesgue measure. This is the simple fact behind Lemma 11 below.

Let Z be equal to the number of x-bubbles, G, contained in R of diameter $\leqslant m'x^2$ and of area $\geqslant 2^{-2N_0}x^4$ and so that there does not exist an (i, j) such that G contributes to $X_{i,j}^K$.

Lemma 11. $E[Z] \leqslant \dfrac{2\delta}{10^8|x|^3}|R|$ *for x small.*

Proof. By Lemma 10, it will suffice to consider bubbles that do not contribute to $X_{i,j}$ for any (i,j).

Let $D = R/\cup_{i,j} \Delta'_{i,j}$. Then if a bubble is not within $\Delta'_{i,j}$ and is of diameter less than $m'x^2$, then it must be completely contained in $D^{m'x^2}$, the $m'x^2$ envelope of D. The Lebesgue measure of $D^{m'x^2}$ is readily seen to be bounded by

$$8\,\frac{m'x^2+\gamma m x^2}{mx^2}\,|R| \leqslant \left(8\gamma + \frac{8m'}{m}\right) < 2^{-2N_0}\varepsilon^2\delta^2|R|/10^8$$

by our choice of m and γ. Now the expectation of

$$\int_{D^{m'x^2}} I_{W(t,s)\in(0,2x)}\,\mathrm{d}s\,\mathrm{d}t$$

is bounded by $(x/\varepsilon)|D^{m'x^2}| \leqslant 2^{-2N_0}\delta^2|R|x/10^8$ since by our restriction to $(\varepsilon,1]^2$ the density at any point of $W(\mathbf{t}) \leqslant 1/(2\varepsilon)$. Consequently the expectation of the number of such x-bubbles (necessarily of area at least $2^{-2N_0}x^4$) is bounded by $\delta|R|/(10^8x^3)$. □

Proposition 2. *As x tends to zero,*

$$x^6 E\left[\left(\sum X_{ij}^K - g^\gamma\big(W(t_i,s_j)/x,m\big)I_{|W(t_i,s_j)|\leqslant Kx}\right)^2\right] \longrightarrow 0.$$

Proof. Note that X_{ij}^K and $g^\gamma(|W(t_i,s_j)|/x,m)$ are bounded. The expression of interest is equal to

$$\begin{aligned}
&x^6\sum_{i,j} E\left[\left(X_{ij}^K - g^\gamma\big(W(t_i,s_j)/x,m\big)I_{|W(t_i,s_j)|\leqslant Kx}\right)^2\right]\\
&\qquad + x^6\sum_{i,j,k} E\Big[\Big(X_{ij}^K - g^\gamma\big(W(t_i,s_j)/x,m\big)I_{|W(t_i,s_j)|\leqslant Kx}\Big)\\
&\qquad\qquad \times\Big(X_{ik}^K - g^\gamma\big(W(t_i,s_k)/x,m\big)I_{|W(t_i,s_k)|\leqslant Kx}\Big)\Big]\\
&\qquad + x^6\sum_{i,j,k} E\Big[\Big(X_{ij}^K - g^\gamma\big(W(t_i,s_j)/x,m\big)I_{|W(t_i,s_j)|\leqslant Kx}\Big)\\
&\qquad\qquad \times\Big(X_{kj}^K - g^\gamma\big(W(t_k,s_j)/x,m\big)I_{|W(t_k,s_j)|\leqslant Kx}\Big)\Big]\\
&\qquad + x^6\sum_{i\neq i',j\neq j'} E\Big[\Big(X_{ij}^K - g^\gamma\big(W(t_i,s_j)/x,m\big)I_{|W(t_i,s_j)|\leqslant Kx}\Big)\\
&\qquad\qquad \times\Big(X_{i'j'}^K - g^\gamma\big(W(t_{i'},s_{j'})/x,m\big)I_{|W(t_{i'},s_{j'})|\leqslant Kx}\Big)\Big]\\
&\quad\leqslant Cx^6\,\frac{1}{x^4}\,x + C'x^6\left(\frac{1}{x^2}\right)^3 x\\
&\qquad + x^6\sum_{i\neq i',j\neq j'} E\Big[\Big(X_{ij}^K - g^\gamma\big(W(t_i,s_j)/x,m\big)I_{|W(t_i,s_j)|\leqslant Kx}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad \times\Big(X^K_{i'j'} - g^\gamma\big(W(t_{i'}, s_{j'})/x, m\big) I_{|W(t_{i'}, s_{j'})| \leqslant Kx}\Big)\Big]\\
&\leqslant C''x + C''' \sup_{i\neq i' j \neq j'} E\Big[\Big(X^K_{ij} - g^\gamma\big(W(t_i, s_j)/x, m\big) I_{|W(t_i, s_j)| \leqslant Kx}\Big)\\
&\qquad\qquad \times\Big(X^K_{i'j'} - g^\gamma\big(W(t_{i'}, s_{j'})/x, m\big) I_{|W(t'_i, s'_j)| \leqslant Kx}\Big)\\
&\qquad\qquad \Big|\ |W(t_i, s_j)| \leqslant Kx,\ |W(t_{i'}, s_{j'})| \leqslant Kx\Big]
\end{aligned}
$$

Thus it remains to prove that the last term on the right tends to zero. If $i < i'$, $j \leqslant j'$, then we clearly have that $X^K_{i'j'}$ is conditionally independent of X^K_{ij}, $W(t_i, s_j)$ given $W(t_{i'}, s_{j'})$ and so the desired conclusion in this case follows directly from 8. So we treat the case $i < i'$, $j > j'$ (the case $i > i'$, $j < j'$, is the same).

We consider $X^K_{i'j'} - g^\gamma(W(t_{i'}, s_{j'}), m) I_{|W(t_{i'}, s_{j'}| \leqslant Kx}$. We condition on:

(i) The white noise in square $[t_i, t_{i+1}] \times [s_{j'}, s_{j'+1}]$;

(ii) $W(t_i, s_j)$ (necessarily $\leqslant Kx$ in magnitude);

(iii) X^K_{ij}.

Now notice that on $[t_{i'} t_{i'+1}] \times [s_{j'}, s_{j'+1}]$

$$
\begin{aligned}
W(t,s) = {} & W(t_{i'}, s_{j'})\\
& + W(t, s_{j'}) - W(t_{i'}, s_{j'}) \qquad (\equiv B_1(t))\\
& + W(t_{i'}, s) - W(t_{i'}, s_{j'}) \qquad (\equiv B_2(s))\\
& + W(t, s) - W(t, s_{j'}) - W(t_{i'}, s) + W(t_{i'}, s_{j'}) \qquad (\equiv W_3(s,t))
\end{aligned}
$$

B_1, B_2, W_3 are independent. (B_1, W_3) is in addition, independent of (i), (ii) and (iii) above. $B_2(t)$ can be written as:

a) $W(t_i, s) - W(t_i, s_{j'}) \qquad +$

b) $W(t_{i+1}, s) - W(t_{i+1}, s_{j'}) - W(t_i, s) + W(t_i, s_{j'}) \qquad +$

c) $W(t_{i'}, s) - W(t_{1'}, s_{j'}) - W(t_{i+1}, s) + W(t_{i+1}, s_{j'})$

Now these three processes are independent c) is independent of a) b) (i), (ii) and (iii)). b) is (with probability tending to one as $x \to 0) \leqslant |x|^{3/2}$ in supremum norm while given (i), (ii), (iii) a) is converging in distribution to a speed t_i Brownian motion, independent of B_1.

The result now follows by Theorem 2.1 and the boundedness of random variables concerned. □

Proof of Theorem 1. Given Proposition 2 and Lemma 1, we have

$$
\begin{aligned}
& x^3 \Big(\sum X^K_{ij} - g^\gamma(W_{t_i, s_j}/x, m)\Big) \xrightarrow{\text{pr}} 0,\\
& \qquad x^3 \sum g^\gamma(W(t_i, s_j)/x, m) \xrightarrow{\text{pr}} x_1 y_1 \int_R \mathrm{d}L(u,v) \Big(\int_{-K}^{K} g^\gamma(x,m)\,\mathrm{d}x\Big),\\
& \qquad\qquad \text{so} \qquad x^3 \sum X^K_{i,j} \xrightarrow{\text{pr}} x_1 y_1 \int_R \mathrm{d}L(u,v) \Big(\int_{-K}^{K} g^\gamma(x,m)\,\mathrm{d}x\Big).
\end{aligned}
$$

Now, recall that we subdivided $[\varepsilon, 1]^2$ into disjoint rectangles R_i so we reintroduce the suffixes i then

$$x^3 \sum_i \sum_{j,l} (X_{lj}^K)^i \xrightarrow{\mathrm{pr}} c \int_{[\varepsilon,1]^2} st \, \mathrm{d}L(st) + \delta' O\Big(\int_{[0,1]} \mathrm{d}L(st) \Big)$$

for some $c > 0$.

Now if X is equal to the number of x-bubbles contained in $[0,1]^2$ then $X - \sum_\ell \sum (X_{ij}^K)^l$ counts the x-bubbles that

a) are of diameter $> m'x^2$,

b) are of size $\leqslant 2^{-2N_0} x^4$,

c) achieve value x in $[0,\varepsilon] \times [0,1]$ or $[0,1] \times [0,\varepsilon]$,

d) are contained in R_i for some i but not in X_i', for any i,

e) intersect δR_i for some i but are of area $\geqslant 2^{-2N_0} x^4$ and diameter $< m'x^2$.

But we have shown that the expectation of bubbles satisfying a) $\to$ c) is bounded by δ / x^3, the expectation of bubbles satisfying d) $\leqslant C x^3 \delta |R_i| \leqslant C\delta / x^3$ for C not depending on x. Those bubbles satisfying e) have expectation bounded by $C(\delta) M / x^2$ by an argument similar to that used in the proof of Lemma 11. We are done by the arbitrariness of δ. $\square$

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