Necessary and sufficient conditions for the supermartingale property of a stochastic integral with respect to a local martingale

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Summary. This paper deals with necessary and sufficient conditions for the supermartingale property of a stochastic integral with respect to a local martingale. A basic answer is due to Ansel and Stricker [1].

Recently, Schachermayer [3], and Kabanov and Stricker [2] have also dealt with this problem requiring an integrability condition at arbitrary sequences of stopping times (cf. (7)). The subject of this paper is how to improve these results by imposing this integrability condition at a considerably smaller class of stopping times (cf. (3)). As a result it turns out that it suffices to impose this integrability condition at the time horizon and at one particular sequence of hitting times (cf. Theorem 2). By means of a counterexample (cf. Section 1), it is shown that none of the two conditions can be omitted. As a side result we give an application to mathematical finance (cf. Section 3).

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Introduction

Let $T > 0$ be a fixed and finite time horizon and let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{0 \leqslant t \leqslant T})$ be a filtered probability space satisfying the usual conditions. In the sequel, every stopping time under consideration is supposed to be $[0, T] \cup \{+\infty\}$-valued and $(\tau_n)_{n=0}^{\infty}$ always denotes a sequence of stopping times.

Let S be a local martingale and let $H \in L(S)$, where $L(S)$ denotes the set of S-integrable, predictable processes. The purpose of the present paper

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is to discuss necessary and sufficient conditions so that the stochastic integral $X := H \cdot S$ is a supermartingale. A familiar answer to this question is given by a corollary to Proposition 3.3 of Ansel and Stricker [1].

Theorem 1. *Let S be a local martingale and let H be an S-integrable, predictable process. The stochastic integral $H \cdot S$ is a supermartingale iff $(H \cdot S)^-$ is dominated by a martingale.*

The necessity part is trivial. The proof of the converse is similar to the proof of Corollaire 3.5 by Ansel and Stricker [1]. Moreover, the conditions of Theorem 1 imply that the stochastic integral $H \cdot S$ is also a local martingale, cf. Ansel and Stricker [1], Proposition 3.3.

The purpose of the present paper is to replace the existence of a dominating martingale for $(H \cdot S)^-$ in Theorem 1 by an integrability condition and to give alternative characterizations of the supermartingale property of the stochastic integral $H \cdot S$.

Let us briefly explain the basic idea behind our approach. In case $X^- = (H \cdot S)^-$ is uniformly bounded, Theorem 1 clearly implies the supermartingale property of the stochastic integral $X = H \cdot S$. This motivates the question whether it is sufficient to presuppose that the sets $\{X = H \cdot S \leqslant -n\}$ become small as $n \to \infty$. For this purpose, define the sequence $(\sigma_n)_{n=0}^{\infty}$ of hitting times by

$$\sigma_0 := 0 \qquad \text{and} \qquad \sigma_n := \inf\{0 \leqslant t \leqslant T : X_t \leqslant -n\}, \quad n \geqslant 1, \tag{1}$$

and consider

$$\varliminf_{n \to \infty} \mathbb{E}\big[X_{\sigma_n}^- I_{\{\sigma_n < \infty\}}\big] = 0. \tag{2}$$

Note that $\sigma_n \uparrow \infty$ a.s. and thus our condition (2) is a special case of condition (19) used by Schachermayer [3], Lemma 1. A detailed discussion on the relation between our results and the results by Schachermayer [3], and Kabanov and Stricker [2] can be found below, cf. Corollary 1, and in Section 2. In the sequel we discuss the role of condition (2) in connection with the supermartingale property of the stochastic integral $X = H \cdot S$.

This paper contains the following results:

In Section 1, we present a counterexample showing that condition (2) alone is not a sufficient condition for the supermartingale property of the stochastic integral $X = H \cdot S$. This counterexample even shows that condition (2) does not imply the integrability of X_T^-.

Moreover, we supplement condition (2) with an integrability condition in order to obtain a criterion for the supermartingale property of the stochastic integral $X = H \cdot S$. We will present two versions of this criterion, the detailed proofs being postponed to Section 2.

Theorem 2. *Let S be a local martingale and let H be an S-integrable predictable process. Moreover, let $(\sigma_n)_{n=0}^{\infty}$ be the sequence of hitting times according to (1). The stochastic integral $X = H \cdot S$ is a supermartingale iff*

$$\lim_{n\to\infty} \mathbb{E}\big[X_{\sigma_n}^- I_{\{\sigma_n<\infty\}}\big] = 0 \qquad \textit{and} \qquad \mathbb{E}[X_T^-] < \infty. \tag{3}$$

Note that in general none of the two conditions in (3) alone implies the supermartingale property of the stochastic integral $X = H \cdot S$. As mentioned above, the counterexample in Section 1 yields that it is not possible to omit the integrability of X_T^-. On the other hand, the well-known construction of the so-called doubling strategy in mathematical finance shows that condition (2) (first condition in (3)) cannot be omitted.

Corollary 1. *Let S be a local martingale and let H be an S-integrable predictable process. The stochastic integral $X = H \cdot S$ is a supermartingale iff*

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\big[X_{\sigma_n}^- I_{\{\sigma_n<\infty\}}\big] = 0 \quad \textit{and} \\ &\lim_{n\to\infty} \mathbb{E}\big[X_{\tau_n}^- I_{\{\tau_n<\infty\}}\big] < \infty, \;\; \textit{whenever} \;\; \tau_n \uparrow \infty \;\; \textit{a.s.} \end{aligned} \tag{4}$$

Recent papers by Schachermayer [3], and by Kabanov and Stricker [2] deal with a similar problem. Their results are implied by Theorem 2 and its Corollary 1, whose conditions are slightly weaker than those of the results by Schachermayer [3] and by Kabanov and Stricker [2]. The necessary tool for proving these implications is the easy Lemma 2, cf. Section 4.

Finally, we give an interesting application to mathematical finance. It turns out that the integrability of X_T^- in (3) can be omitted if Theorem 2 is applied to the no-arbitrage problem. As a result we obtain that the existence of an equivalent local martingale measure yields a broader class of arbitrage-free wealth-processes than those considered in the usual theory.

Theorem 3. *Let S be a local martingale. Then the set*

$$\mathcal{X} := \Big\{X = H \cdot S : H \in L(S) \wedge \lim_{n\to\infty} \mathbb{E}\big[(H \cdot S)_{\sigma_n}^- I_{\{\sigma_n<\infty\}}\big] = 0\Big\}$$

is free of arbitrage.

1 The Counterexample

The idea of the counterexample is based on Schachermayer [3], Section 3.

Define $\Omega := \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ and let $\mathcal{B}$ be the Borel sigma-algebra of Ω. By abuse of notation we denote the components and the coordinate functions by the same symbol, i.e. the elements of Ω are written as $\omega = ((\eta_n)_{n=1}^\infty, (\xi_n)_{n=1}^\infty)$. Fix a finite time horizon T and let $(t_n)_{n=0}^\infty$ be a strictly increasing sequence such that $t_0 = 0$ and $\lim_{n\to\infty} t_n = T$.

Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=1}^\infty$ be two sequences of real numbers given by $a_0 := -(1+\varepsilon)$, $0 < \varepsilon < 1$, and

$$a_n := -\sqrt{n}\,\mathrm{e}^n, \qquad n \geqslant 1$$
$$b_n := (-1)^n\,\mathrm{e}^{n-1}, \qquad n \geqslant 1.$$

Note that $(a_n)_{n=0}^\infty$ is strictly decreasing to $-\infty$ and that $b_n > a_{n-1}$, for all $n \geqslant 1$. Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{B})$ under which the random variables $((\eta_n)_{n=1}^\infty, (\xi_n)_{n=1}^\infty)$ are independent and satisfy

$$\mathbb{P}\{\eta_n = 1\} = \frac{a_{n-1} - a_n}{b_n - a_n} \approx 1 - \mathrm{e}^{-1},$$
$$\mathbb{P}\{\xi_n = 1\} = \frac{1}{n+1} =: p_n.$$

For notational convenience we write

$$d_n := \frac{a_{n-1} - b_n}{a_n - b_n} \approx \mathrm{e}^{-1} \qquad \text{as well as} \qquad e_n = \prod_{j=1}^n d_j \approx \mathrm{e}^{-n}.$$

Similarly as in Schachermayer [3], we define the process $(S_{t_n})_{n=0}^\infty$ inductively by $S_{t_0} = 0$ and

$$n \geqslant 1, \qquad S_{t_n} = \begin{cases} a_n & \text{if } \xi_k = \eta_k = 0 \text{ for } k < n, \text{ and } \eta_n = 0, \\ b_n & \text{if } \xi_k = \eta_k = 0 \text{ for } k < n, \text{ and } \eta_n = 1, \\ S_{t_n-1} & \text{otherwise.} \end{cases}$$

By means of the Lemma of Borel and Cantelli,

$$\sum_{n=1}^\infty \mathbb{P}\{\xi_n = 1\} = \infty$$

implies $\mathbb{P}\{\overline{\lim}_{n\to\infty} \xi_n = 1\} = 1$. Thus, the process $(S_{t_n})_{n=0}^\infty$ becomes almost surely eventually constant and we define the pointwise limit $S_T(\omega) := \lim_{n\to\infty} S_{t_n}(\omega)$.

It is easy to verify that the probabilities $\mathbb{P}\{\eta_n = 1\}$, $n \geqslant 1$, are chosen such that $\mathbb{E}[S_{t_n} \mid S_{t_0}, \dots, S_{t_{n-1}}] = S_{t_{n-1}}$, $n \geqslant 1$. Thus, the continuous time process $(S_t)_{0 \leqslant t < \infty}$ defined by $S_t := S_{t_{n-1}}$ on $[t_{n-1}, t_n[$ is a local martingale with respect to its natural filtration.

Next, we show $\mathbb{E}[S_T^-] = \infty$ which implies that S_T is not a supermartingale. For this purpose we observe $\prod_{j=1}^{n-1}(1 - p_j) = 1/n$ and

$$\begin{aligned} \mathbb{P}\{S_T^- = |a_n|\} &= \mathbb{P}\{\xi_k = \eta_k = 0,\ k < n;\ \xi_n = 1;\ \eta_n = 0\} \\ &= \prod_{j=1}^{n-1}(1 - p_j) p_n e_n \approx \frac{1}{n^2}\,\mathrm{e}^{-n}. \end{aligned} \tag{5}$$

In case n is even, we obviously have $\mathbb{P}\{S_T^- = b_n\} = 0$ since $b_n > 0$. If n is odd, we obtain

$$\begin{aligned}\mathbb{P}\{S_T^- = |b_n|\} &= \mathbb{P}\{\xi_k = \eta_k = 0,\, k < n;\, \eta_n = 1\} \\ &= \prod_{j=1}^{n-1}(1-p_j)e_{n-1}(1-d_n) \approx \frac{1}{n}\,\mathrm{e}^{-(n-1)}.\end{aligned} \tag{6}$$

Combining (5) and (6) yields

$$\begin{aligned}\mathbb{E}[S_T^-] &= \sum_{n=1}^{\infty}|a_n|\,\mathbb{P}\{S_T^- = |a_n|\} + \sum_{n=1}^{\infty}|b_n|\,\mathbb{P}\{S_T^- = |b_n|\} \\ &\approx \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} + \sum_{n=1}^{\infty}\frac{1}{n} = \infty.\end{aligned}$$

Finally, we show that $\underline{\lim}_{n\to\infty}\mathbb{E}[S_{\sigma_n}^- I_{\{\sigma_n<\infty\}}] = 0$, where the sequence $(\sigma_n)_{n=0}^{\infty}$ is defined by $\sigma_0 := 0$ and $\sigma_n := \inf\{t \geqslant 0 : S_t \leqslant -n\}$, $n \geqslant 1$. For this purpose define

$$k_n := \inf\Big\{k \in \mathbb{N} : -\sqrt{k}\,\mathrm{e}^k \leqslant -n\Big\}, \qquad n \geqslant 1.$$

and observe that the sequence $(k_n)_{n=1}^{\infty}$ is increasing to $+\infty$. Note that $\{\exists k : S_{t_k} \leqslant -n\} = \{S_{t_{k_n}} = a_{k_n}\}$. Indeed, the existence of some k such that $S_{t_k} \leqslant -n$ implies that at least one element of the sequence $a_1, \ldots, a_{k-1}, a_k, b_k$ is less than $-n$. This can only happen if $S_{t_{k_n}} = a_{k_n}$. The reverse inclusion $\{S_{t_{k_n}} = a_{k_n}\} \subseteq \{\exists k : S_{t_k} \leqslant -n\}$ is obvious and therefore

$$\begin{aligned}\mathbb{P}\{\sigma_n < \infty\} &= \mathbb{P}\{\exists k : S_{t_k} \leqslant -n\} \\ &= \mathbb{P}\{S_{t_{k_n}} = a_{k_n}\} = \mathbb{P}\{\xi_k = \eta_k = 0,\, k < k_n;\, \eta_{k_n} = 0\} \\ &= \prod_{j=1}^{k_n-1}(1-p_j)e_{k_n} \approx \frac{1}{k_n}\,\mathrm{e}^{-k_n}.\end{aligned}$$

Hence, we obtain

$$\underline{\lim}_{n\to\infty}\mathbb{E}\big[S_{\sigma_n}^- I_{\{\sigma_n<\infty\}}\big] \leqslant \lim_{n\to\infty}\sqrt{k_n}\,\mathrm{e}^{k_n}\,\mathbb{P}\{\sigma_n < \infty\} = \lim_{n\to\infty}\frac{1}{\sqrt{k_n}} = 0.$$

Summing up, this example shows that the validity of condition (2) alone is not a sufficient condition for the supermartingale property of the stochastic integral $H \cdot S$. □

2 Detailed Results

Let us begin with the proofs of Theorem 2 and Corollary 1. For the reader's convenience, we repeat the assertions here.

Theorem 2. *Let S be a local martingale and let H be an S-integrable predictable process. The stochastic integral $X = H \cdot S$ is a supermartingale iff*

$$\varliminf_{n\to\infty} \mathbb{E}\big[X^-_{\sigma_n} I_{\{\sigma_n<\infty\}}\big] = 0 \qquad \textit{and} \qquad \mathbb{E}[X^-_T] < \infty. \tag{3}$$

Proof. The necessity part is an immediate consequence of Theorem 1. Concerning the sufficiency part, we proceed as follows: using similar arguments as in Schachermayer [3], proof of Lemma 1, we obtain that the first condition in (3) implies the local supermartingale property of $X = H \cdot S$. This together with the integrability of X^-_T then implies the supermartingale property of $X = H \cdot S$.

Condition (3) implies the existence of a subsequence $(\sigma_n)_{n\in\mathbb{N}_1}$, $\mathbb{N}_1 \subseteq \mathbb{N}$, such that

$$\lim_{n\in\mathbb{N}_1} \mathbb{E}\big[X^-_{\sigma_n} I_{\{\sigma_n<\infty\}}\big] = 0.$$

Hence, $\lim_{n\in\mathbb{N}_1}\mathbb{E}[X^-_{\sigma_n} I_{\{\sigma_n<\infty\}} \,|\, \mathcal{F}_t] = 0$ in probability and thus there exists another subsequence $(\sigma_n)_{n\in\mathbb{N}_2}$, $\mathbb{N}_2 \subseteq \mathbb{N}_1$, such that

$$\lim_{n\in\mathbb{N}_2} \mathbb{E}\big[X^-_{\sigma_n} I_{\{\sigma_n<\infty\}} \,\big|\, \mathcal{F}_t\big] = 0 \qquad \text{a.s.}$$

We first verify the local supermartingale property of X along the lines of Schachermayer [3], proof of Lemma 1. For this purpose observe that the first condition in (3) implies for sufficiently large n that $\mathbb{E}[\theta_n] < \infty$, where $\theta_n := \max(X^-_{\sigma_n} I_{\{\sigma_n<\infty\}}, n)$, $n \in \mathbb{N}_2$. Since $\inf_{0\leqslant t\leqslant T} X_{t\wedge\sigma_n} \geqslant -n$ on $\{\sigma_n = \infty\}$ and since $\inf_{0\leqslant t\leqslant T} X_{t\wedge\sigma_n} \geqslant -X_{\sigma_n} I_{\{\sigma_n<\infty\}}$ on $\{\sigma_n < \infty\}$, we have $X^{\sigma_n} \geqslant -\theta_n$ a.s., $n \in \mathbb{N}_2$. By means of Theorem 1 we obtain that X^{σ_n} is a supermartingale, $n \in \mathbb{N}_2$. Thus, we have $-X_{t\wedge\sigma_n} \leqslant \mathbb{E}[-X_{T\wedge\sigma_n} \,|\, \mathcal{F}_t]$ and therefore $X^-_{t\wedge\sigma_n} \leqslant \mathbb{E}[X^-_{T\wedge\sigma_n} \,|\, \mathcal{F}_t]$, $n \in \mathbb{N}_2$. In particular we obtain

$$\begin{aligned}
X^-_t &= \lim_{n\in\mathbb{N}_2} X^-_{t\wedge\sigma_n} \\
&\leqslant \varlimsup_{n\in\mathbb{N}_2} \mathbb{E}[X^-_{T\wedge\sigma_n} \,|\, \mathcal{F}_t] \\
&\leqslant \mathbb{E}[X^-_T \,|\, \mathcal{F}_t] + \lim_{n\in\mathbb{N}_2} \mathbb{E}\big[X^-_{\sigma_n} I_{\{\sigma_n<\infty\}} \,\big|\, \mathcal{F}_t\big] = \mathbb{E}[X^-_T \,|\, \mathcal{F}_t] \qquad \text{a.s..}
\end{aligned}$$

Thus $X_t \geqslant -X^-_t \geqslant -\mathbb{E}[X^-_T \,|\, \mathcal{F}_t]$ and by means of Theorem 1 we obtain that X is a supermartingale. □

Corollary 1. *Let S be a local martingale and let H be an S-integrable predictable process. The stochastic integral $X = H \cdot S$ is a supermartingale iff*

$$\begin{aligned}
&\varliminf_{n\to\infty} \mathbb{E}\big[X^-_{\sigma_n} I_{\{\sigma_n<\infty\}}\big] = 0 \quad \textit{and} \\
&\varliminf_{n\to\infty} \mathbb{E}\big[X^-_{\tau_n} I_{\{\tau_n<\infty\}}\big] < \infty, \quad \textit{whenever} \quad \tau_n \uparrow \infty \textit{ a.s..}
\end{aligned} \tag{4}$$

Proof. The necessity part is an immediate consequence of Theorem 1. The converse is easy to verify by means of Lemma 2. □

Now, it is clear that Lemma 1 in Schachermayer [3] is an immediate consequence of Corollary 1. For the reader's convenience, we repeat the result by Schachermayer [3].

Lemma 1. *Let S be a local martingale and let H be an S-integrable, predictable process. The stochastic integral $X = H \cdot S$ is a supermartingale iff*

$$\varliminf_{n\to\infty} \mathbb{E}\big[X^-_{\tau_n} I_{\{\tau_n<\infty\}}\big] = 0, \qquad \textit{whenever } \tau_n \uparrow \infty \textit{ a.s.} \tag{7}$$

By means of our counterexample, cf. Section 1, we obtain that in Schachermayer [3], Lemma 1, it is not possible to replace the arbitrary sequences $(\tau_n)_{n=0}^\infty$ of stopping times in condition (7) by the sequence $(\sigma_n)_{n=0}^\infty$ of hitting times defined in (1) without imposing further conditions.

In a recent paper, Kabanov and Stricker [2], Lemma 1, give an alternative proof of the result by Schachermayer [3], isolating the second part of Schachermayer's original proof as an assertion about nonnegative local submartingales. Similar to our Theorem 2, one can state and prove an assertion in the spirit of Kabanov and Stricker [2] in the following way.

Theorem 4. *Let Y be a nonnegative local submartingale. Then Y is a submartingale iff*

$$\varliminf_{n\to\infty} \mathbb{E}\big[Y_{\sigma_n} I_{\{\sigma_n<\infty\}}\big] = 0 \qquad \textit{and} \qquad \mathbb{E}[Y_T] < \infty. \tag{8}$$

3 Application to Finance

The aim of this section is to apply our results to mathematical finance. For this purpose, we assume throughout this section that S is a d-dimensional local martingale with $S^1 \equiv 1$ and that $H \in L(S)$ is an S-integrable and predictable process.

Recall that a wealth-process $X = H \cdot S$ is said to satisfy the no-arbitrage condition, if

$$X_0 = 0 \text{ a.s.}, \quad X_T \geqslant 0 \text{ a.s.} \qquad \Longrightarrow \qquad X_T = 0 \text{ a.s.} \tag{NA}$$

It is well-known that the supermartingale property of $X = H \cdot S$ is a sufficient condition for a wealth-process to be free of arbitrage. Theorem 1 implies that the set

$$\widetilde{\mathcal{X}} := \big\{X = H \cdot S : H \in L(S) \wedge (H \cdot S)^- \text{ is dominated by a martingale}\big\}$$

satisfies the NA-condition. Usually, one defines

$$\mathcal{X}_a := \big\{X = H \cdot S : H \in L(S) \wedge H \cdot S \geqslant -a \text{ a.s.}\big\}, \qquad a \in \mathbb{R},$$

and thus the set $\bigcup_{a\in\mathbb{R}} \mathcal{X}_a$ of wealth-processes uniformly bounded from below is free of arbitrage, since $\bigcup_{a\in\mathbb{R}} \mathcal{X}_a \subseteq \widetilde{\mathcal{X}}$. Define

$$\mathcal{X} := \Big\{ X = H\cdot S : H \in L(S) \wedge \varliminf_{n\to\infty} \mathbb{E}\big[(H\cdot S)^-_{\sigma_n} I_{\{\sigma_n<\infty\}}\big] = 0 \Big\}$$

and observe that $\bigcup_{a\in\mathbb{R}} \mathcal{X}_a$ is a proper subset of $\mathcal{X}$. By means of our counterexample, we even obtain that $\widetilde{\mathcal{X}}$ is a proper subset of $\mathcal{X}$. Since Theorem 3 implies the no-arbitrage property for the set $\mathcal{X}$, we obtain a broader class of arbitrage-free wealth-processes than those which are considered in the usual theory.

Theorem 3. *Let S be a local martingale. Then $\mathcal{X}$ is free of arbitrage.*

Proof. Let $X = H\cdot S$ be a wealth-process satisfying $X_0 = 0$ a.s. and $X_T \geqslant 0$ a.s.. Thus, we obviously have $\mathbb{E}[X_T^-] < \infty$ and by means of Theorem 2 we obtain the supermartingale property of $X = H\cdot S$. Hence, $\mathbb{E}[X_T] \leqslant \mathbb{E}[X_0] = 0$ implies $X_T = 0$ a.s. and the assertion is verified. □

4 Appendix

This section contains an easy auxiliary lemma stating an integrability condition for nonnegative adapted processes with càdlàg paths. Using an argument of Kabanov and Stricker [2], Lemma 1, we present a proof for the reader's convenience.

Lemma 2. *Let Y be a nonnegative adapted process with càdlàg paths. Define*

$$\tau_n := \begin{cases} T & \text{if there exists } t \leqslant T \text{ such that } Y_t \geqslant n \\ \infty & \text{otherwise} \end{cases} \tag{9}$$

and assume

$$\varliminf_{n\to\infty} \mathbb{E}[Y_{\tau_n} I_{\{\tau_n<\infty\}}] < \infty. \tag{10}$$

Then $\mathbb{E}[Y_T] < \infty$.

Proof. Condition (10) implies the existence of a subsequence $(\tau_n)_{n\in\mathbb{N}_1}$, $\mathbb{N}_1 \subseteq \mathbb{N}$, such that $\lim_{n\in\mathbb{N}_1} \mathbb{E}[Y_{\tau_n} I_{\{\tau_n<\infty\}}] < \infty$. Hence

$$\mathbb{E}[Y_T] \leqslant n\,\mathbb{P}\{\tau_n = \infty\} + \mathbb{E}\big[Y_{\tau_n} I_{\{\tau_n<\infty\}}\big] < \infty,$$

for sufficiently large $n \in \mathbb{N}_1$ and the assertion is proved. □

Note that the counterexample in Section 1 yields that in Lemma 2 it is not possible to replace the sequence of stopping times $(\tau_n)_{n=0}^\infty$ in condition (9) by the sequence of hitting times $(\sigma_n)_{n=0}^\infty$ defined in (1).

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