On the maximum of a diffusion process in a random Lévy environment

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Introduction

In [5], Carmona–Petit–Yor investigated asymptotic behaviour of the tail of the distribution of the maximum of a diffusion process in a random Lévy environment. This problem is a diffusion analogue of Afanas'ev [1] and a generalization of Kawazu–Tanaka [11]. In this paper we attempt to complete a result in [5].

Following [5], we consider $(X_t;\, t \geqslant 0)$ and $(\xi_t;\, t \geqslant 0)$ independent Lévy processes starting from zero and admitting first moments such that $-\infty \leqslant \mathbb{E}[X_1] < 0$ and $-\infty \leqslant \mathbb{E}[\xi_1] < 0$. Set

$$V(x) = \begin{cases} X_x & \text{if } x \geqslant 0, \\ -\xi_{-x} & \text{if } x \leqslant 0, \end{cases}$$

as a random environment. Given a sample function V, let $(\Xi(t, V);\, t \geqslant 0)$ be a diffusion process starting from zero with generator

$$\frac{1}{2}\,\mathrm{e}^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathrm{e}^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\Big).$$

When V is considered to be random, the process $(\Xi(t);\, t \geqslant 0)$ is called a diffusion process in a random Lévy environment. Let $\mathcal{P}$ be the full probability of Ξ. Since the scale function of $\Xi(\,.\,, V)$ is $x \mapsto \int_0^x \mathrm{e}^{-V(y)}\,\mathrm{d}y$, we have, for $x > 0$,

$$\mathcal{P}(x) := \mathcal{P}\Big\{\max_{t\geqslant 0} \Xi(t) > x\Big\} = \mathbb{E}\bigg[\frac{A}{A + A_x}\bigg], \tag{1}$$

where

$$A = \int_0^\infty \mathrm{e}^{\xi_t}\,\mathrm{d}t \qquad \text{and} \qquad A_x = \int_0^x \mathrm{e}^{-X_t}\,\mathrm{d}t.$$

We know that $\max_{t\geqslant 0} \Xi(t)$ is finite $\mathcal{P}$ almost surely because $\mathbb{E}[X_1]$ and $\mathbb{E}[\xi_1]$ are in $[-\infty, 0)$. Our basic concern is to determine the rate of decay of $\mathcal{P}(x)$ as $x \to \infty$.

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To state our results we need the following. The Laplace exponent ϕ of X is defined by

$$\mathbb{E}\left[\mathrm{e}^{\theta X_t}\right] = \mathrm{e}^{-t\phi(\theta)}, \qquad t \geqslant 0,\, \theta \in \mathbb{R}.$$

Denote by ψ the Laplace exponent of ξ. Our study in this paper was motivated by the following result.

Theorem 0 (Carmona–Petit–Yor [5]). *Assume that:*

(a) *ϕ is defined in a neighbourhood of* 1*;*
(b) *ψ is defined in a neighbourhood of* 1*, and* $\psi(1) > 0$*.*

1) *If $\phi'(1) > 0$, then as $x \to \infty$,*

$$\mathcal{P}(x) \sim \mathrm{e}^{-x\phi(1)}\phi'(1)\psi(1)^{-1}.$$

2) *If $\phi'(1) = 0$ and $\phi''(1) < 0$, then*

$$\mathcal{P}(x) \sim \mathrm{e}^{-x\phi(1)}\psi(1)^{-1}\sqrt{|\phi''(1)|/2\pi x}.$$

3) *If $\phi'(1) < 0$, then*

$$\mathcal{P}(x) = o(\mathrm{e}^{-x\phi(1)}).$$

Since ϕ is concave, $\phi(1) > 0$ if $\phi'(1) \geqslant 0$. But it may occur that $\phi(1) \leqslant 0$ if $\phi'(1) < 0$. In fact we have a typical example $X_t = B_t - \alpha t$ where B is a Brownian motion and $0 < \alpha \leqslant 2^{-1}$. Namely, 3) of Theorem 0 does not always tell us good information. One of our aims is to improve 3) of Theorem 0.

Now let us state our results. Each result below is proved under all (or some) of the following conditions.

(c) *There exists $\alpha \in (0,1)$ such that ϕ is defined in a neighbourhood of α, and $\phi'(\alpha) = 0$.*
(d) *ψ is defined in a neighbourhood of α, and $\psi(\alpha) > 0$.*
(e) *X is not of the form $X_t = bt + \check{X}_t$ where $b \neq 0$ and $\check{X}$ is a compound Poisson process which takes values in $r\mathbb{Z}$ with some $r > 0$.*

Our main result in this paper is

Theorem 1. *Let the conditions* (c), (d) *and* (e) *be satisfied. Then as $x \to \infty$,*

$$\mathcal{P}(x) \sim Cx^{-3/2}\exp\bigl(-x\phi(\alpha)\bigr)$$

with

$$C = \frac{c_1}{\sqrt{2\pi|\phi''(\alpha)|}}\int_0^\infty\!\!\int_{-\infty}^\infty \mathrm{e}^{-\alpha x} g_\lambda(0)\bar{g}_\lambda(x)\,\mathbb{E}\left[A\,\mathrm{e}^{-\lambda A}\right]\mathrm{d}x\,\mathrm{d}\lambda \in (0,\infty),$$

where

$$c_1 = \exp\biggl(\int_0^\infty (\mathrm{e}^{-t}-1)t^{-1}\mathrm{e}^{t\phi(\alpha)}\,\mathbb{P}\{X_t = 0\}\,\mathrm{d}t\biggr),$$

and $g_\lambda(0)$ and $\bar{g}_\lambda(x)$ are given by (5) *in Section 3.*

The rate of decay in Theorem 1 is compatible with the previous works [1] and [11]. This theorem is based on the following estimates.

Proposition 1 (Upper bound). *Assume the conditions* (c) *and* (d). *Then there exists $C_1 < \infty$ such that, for any $x > 0$,*

$$\mathcal{P}(x) \leqslant C_1 x^{-3/2} \mathrm{e}^{-x\phi(\alpha)}.$$

Proposition 2 (Lower bound). *Assume the conditions* (c) *and* (e). *Then we have*

$$\liminf_{x\to\infty} \mathrm{e}^{x\phi(\alpha)} x^{3/2} \mathcal{P}(x) \geqslant C > 0,$$

where C is as in Theorem 1.

We observe that (a) with $\phi'(1) < 0$ implies (c), and that (b) implies (d). Hence Proposition 1 is an extention of 3) of Theorem 0. If we consider the natural environment, i.e., $\phi \equiv \psi$, then (d) is not needed. The restriction similar to (e) has already appeared in the discrete time case studied by Afanas'ev [1]. When the environment is made up of Brownian motions with negative drift, we compute the precise value of C by the different manner from Kawazu–Tanaka [11]. In Appendix, 1) and 2) of Theorem 0 will be considered.

1 Preliminaries

For the Lévy process X, we set $M_t = \sup_{0\leqslant s\leqslant t} X_s$. Let σ_k be the first hitting time of $(-\infty, -k]$, $k \geqslant 0$. That is,

$$\sigma_k = \inf\{t > 0 : X_t \leqslant -k\}.$$

Quantities related to the dual process $\bar{X} := -X$ are denoted by bars. For example $\bar{M}$, $\bar{\sigma}_0$ and so forth. When (c) is satisfied, we define the new probability $\hat{\mathbb{P}}$, called the Girsanov (or Esscher) transform of $\mathbb{P}$, as follows:

$$\hat{\mathbb{P}} = \mathrm{e}^{\alpha X(t) + t\phi(\alpha)} \cdot \mathbb{P} \qquad \text{on } \mathcal{F}_t := \mathcal{F}(X_s : 0 \leqslant s \leqslant t).$$

This relation also holds if the fixed time t is replaced by an $\mathcal{F}_t$ stopping time assumed finite under both $\mathbb{P}$ and $\hat{\mathbb{P}}$. Put $\gamma = \mathrm{e}^{-\phi(\alpha)}$. Under $\hat{\mathbb{P}}$, the process X is a Lévy process with Laplace exponent $\phi(\cdot + \alpha) - \phi(\alpha)$. Denote by $(\tau_s;\, s \geqslant 0)$ the right continuous inverse of a local time process of $M - X$ at 0. Note that local time is defined even if 0 is not regular for $\{0\}$ as in Fristedt [9]. In this paper we select particular normalization factors in local times such that $-\log \mathbb{E}[\mathrm{e}^{-\tau_1}] = 1$. Set $M_{\tau(s)} = \infty$ if $\tau_s = \infty$. For the subordinator $(M_{\tau(s)};\, s \geqslant 0)$, we introduce

$$U(x) = \int_0^\infty \hat{\mathbb{P}}\{M_{\tau(s)} < x\}\, \mathrm{d}s, \qquad \bar{U}(x) = \int_0^\infty \hat{\mathbb{P}}\{\bar{M}_{\bar{\tau}(s)} < x\}\, \mathrm{d}s.$$

That is, U is the left limit of the renewal function associated with the ladder height process of the Lévy process with Laplace exponent $\phi(\cdot+\alpha)-\phi(\alpha)$, and $\bar{U}$ is its dual. We define time homogeneous Markov processes Y and $\bar{Y}$ on $(0,\infty)$ whose transition functions are given by

$$\mathbb{P}_x\{Y_t \in \mathrm{d}y\} = \frac{\bar{U}(y)}{\bar{U}(x)}\hat{\mathbb{P}}_x\{X_t \in \mathrm{d}y,\, \sigma_0 > t\}, \qquad x>0,$$
$$\mathbb{P}_x\big\{\bar{Y}_t \in \mathrm{d}y\big\} = \frac{U(y)}{U(x)}\hat{\mathbb{P}}_x\big\{\bar{X}_t \in \mathrm{d}y,\, \bar{\sigma}_0 > t\big\}, \qquad x>0.$$

By the definition of $\hat{\mathbb{P}}$, $\hat{\mathbb{E}}[X_1] = -\phi'(\alpha) = 0$ and $0 < \hat{\mathbb{E}}[X_1^2] = -\phi''(\alpha) < \infty$, so that X oscillates under $\hat{\mathbb{P}}$. Hence Y and $\bar{Y}$ are conservative, see e.g. [2, p. 184]. Let $\mathbb{D}[0,s]$ be the space of càdlàg functions on $[0,s]$ endowed with Skorohod's topology.

We mention the classification of Lévy processes introduced in [10]. If X is not linear, then X belongs to one of the following classes.

Class I. *For any* $\theta \neq 0$, $\big|\mathbb{E}[\mathrm{e}^{i\theta X_1}]\big| < 1$.

Class II. *The Lévy process X is expressed as* $X_t = bt + \check{X}_t$ *where* $b \neq 0$ *and* $\check{X}$ *is a compound Poisson process which takes values in* $r\mathbb{Z}$ *with some* $r>0$.

Class III. *The Lévy process X is a compound Poisson process which takes values in* $r\mathbb{Z}$ *with some* $r>0$.

If X is in Class II or III, r is the maximal span of the Lévy measure of X. This classification can be derived from the Lévy–Khintchine formula of the characteristic exponent of X. By this classification, (e) is satisfied if and only if X does not belong to Class II, i.e., X is in either Class I or III. In this paper we often assume that X is in Class I because similar arguments work for X in Class III.

2 The Upper Bound

We assume the condition (c) up to Section 4. In this section we assume also the condition (d). So we may choose $\beta \in (\alpha, 1)$ such that $\phi(\beta) > 0$ and $\psi(\beta) > 0$. We fix this β throughout this paper. The following lemma is very easy to prove, but useful.

Lemma 1. *Let* $\phi(\theta)$ (*resp.* $\psi(\theta)$) *be finite for some* $\theta > 0$. *Then* $\mathbb{E}[\mathrm{e}^{\theta M_1}] < \infty$ (*resp.* $\mathbb{E}[\mathrm{e}^{\theta \overline{\xi}_1}] < \infty$, *where* $\overline{\xi}_1 = \sup_{0\leqslant t\leqslant 1} \xi_t$).

Proof. Set $a = \theta/2$ and $b = 0 \vee \phi(a)$. Then $(\mathrm{e}^{aX_t+bt};\, t \geqslant 0)$ is a positive submartingale with respect to $(\mathcal{F}_t;\, t \geqslant 0)$. Since $aM_1 \leqslant \sup_{0\leqslant t\leqslant 1}\{aX_t + bt\}$, by Doob's L^2 martingale inequality

$$\mathbb{E}\big[\mathrm{e}^{2aM_1}\big] \leqslant \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant 1} \mathrm{e}^{aX_t+bt}\bigg]^2 \leqslant 4\,\mathbb{E}\big[\mathrm{e}^{2aX_1+2b}\big].$$

Thus $\mathbb{E}[\mathrm{e}^{\theta M_1}] \leqslant 4\,\mathrm{e}^{[2b-\phi(\theta)]}$. The lemma is proved. □

Recall (1). Since A and A_t are independent,

$$\mathcal{P}(t) = \mathbb{E}[f(A_t)] \qquad \text{where} \qquad f(x) = \mathbb{E}\bigg[\frac{A}{A+x}\bigg]. \tag{2}$$

As for the function $f(x)$, we have the following.

Lemma 2. *There exists $c_2 < \infty$ such that $f(x) \leqslant c_2 x^{-\beta}$ for $x > 0$.*

Proof. By virtue of $\beta \in (0,1)$,

$$A^\beta = \bigg(\sum_{n=0}^{\infty} \int_n^{n+1} \mathrm{e}^{\xi_t}\,\mathrm{d}t\bigg)^\beta \leqslant \sum_{n=0}^{\infty} \mathrm{e}^{\beta\xi_n} \bigg(\int_n^{n+1} \mathrm{e}^{\xi_t-\xi_n}\,\mathrm{d}t\bigg)^\beta.$$

The process $(\xi_{t+n} - \xi_n;\, t \geqslant 0)$ is independent of ξ_n, and have the same law as $(\xi_t;\, t \geqslant 0)$. Hence

$$\mathbb{E}\big[A^\beta\big] \leqslant \sum_{n=0}^{\infty} \mathbb{E}\big[\mathrm{e}^{\beta\xi_n}\big]\,\mathbb{E}\bigg[\bigg(\int_0^1 \mathrm{e}^{\xi_t}\,\mathrm{d}t\bigg)^\beta\bigg] \leqslant \frac{\mathbb{E}\big[\mathrm{e}^{\beta\overline{\xi}_1}\big]}{1-\mathrm{e}^{-\psi(\beta)}} < \infty.$$

The last finiteness follows from Lemma 1. Using the above, we have

$$f(x) = \mathbb{E}\bigg[\frac{A}{A+x}\bigg] \leqslant \mathbb{E}\bigg[\bigg(\frac{A}{A+x}\bigg)^\beta\bigg] \leqslant \mathbb{E}\big[A^\beta\big]x^{-\beta}.$$

The proof of the lemma is complete. □

Lemma 3. *It holds that*

$$\hat{\mathbb{E}}\bigg[(1+|X_1|)\bigg(\int_0^1 \mathrm{e}^{-X_t}\,\mathrm{d}t\bigg)^{-\beta}\bigg] < \infty.$$

Proof. Fix $p \in (1, \beta^{-1})$, and let $p^{-1} + q^{-1} = 1$. By Hölder's inequality,

$$\hat{\mathbb{E}}\bigg[(1+|X_1|)\bigg(\int_0^1 \mathrm{e}^{-X_t}\,\mathrm{d}t\bigg)^{-\beta}\bigg] \leqslant \hat{\mathbb{E}}\bigg[\bigg(\int_0^1 \mathrm{e}^{-X_t}\,\mathrm{d}t\bigg)^{-\beta p}\bigg]^{1/p} \hat{\mathbb{E}}\big[(1+|X_1|)^q\big]^{1/q}$$

The second term in the right hand side is finite because the Laplace transform of X_1 under $\hat{\mathbb{P}}$ exists in a neighbourhood of the origin. If $T = \inf\{t > 0 : X_t > 1\}$, then $\int_0^1 \mathrm{e}^{-X_t}\,\mathrm{d}t \geqslant \mathrm{e}^{-1}(1 \wedge T)$. Putting $b = \beta p$, we have

$$\hat{\mathbb{E}}\left[\left(\int_0^1 \mathrm{e}^{-X_t}\,\mathrm{d}t\right)^{-b}\right] \leqslant \hat{\mathbb{E}}[\mathrm{e}^b;\, T > 1] + \hat{\mathbb{E}}[\mathrm{e}^b T^{-b};\, T \leqslant 1]$$
$$\leqslant \mathrm{e}^b\big(1 + \hat{\mathbb{E}}[T^{-b};\, T \leqslant 1]\big).$$

It is enough to show that $\hat{\mathbb{E}}[T^{-b};\, T \leqslant 1] < \infty$. Applying the integration by parts formula, we have

$$\hat{\mathbb{E}}[T^{-b};\, T \leqslant 1] = \hat{\mathbb{P}}\{T \leqslant 1\} - \lim_{t\downarrow 0} \frac{\hat{\mathbb{P}}\{T \leqslant t\}}{t^b} + b\int_0^1 \frac{1}{t^{b+1}}\hat{\mathbb{P}}\{T \leqslant t\}\,\mathrm{d}t.$$

To estimate $\hat{\mathbb{P}}\{T \leqslant t\}$, we use the fact $\{T \leqslant t\} = \{M_t \geqslant 1\}$. Observe that, under $\hat{\mathbb{P}}$, X is a martingale with respect to $(\mathcal{F}_t;\, t \geqslant 0)$. Doob's L^2 martingale inequality gives

$$\hat{\mathbb{E}}\big[M_t^2\big] \leqslant 4\hat{\mathbb{E}}\big[X_t^2\big] = 4\,vt,$$

where $v = \hat{\mathbb{E}}[X_1^2]$. By Chebyshev's inequality and the above,

$$4\,vt \geqslant \hat{\mathbb{E}}\big[M_t^2;\, M_t \geqslant 1\big] \geqslant \hat{\mathbb{P}}\{M_t \geqslant 1\} = \hat{\mathbb{P}}\{T \leqslant t\}.$$

Since $b = \beta p < 1$, the preceding relations allow us to get

$$\hat{\mathbb{E}}[T^{-b};\, T \leqslant 1] \leqslant 4\,v\Big(1 + \frac{b}{1-b}\Big) = \frac{4\,v}{1-b}.$$

The proof of the lemma is complete. □

Under $\hat{\mathbb{P}}$, the discrete time processes $(X_n;\, n \geqslant 0)$ and $(\bar{X}_n;\, n \geqslant 0)$ are random walks with mean zero and finite variance. In this context, we use the following result given by Vatutin and Dyakonova [13].

Lemma 4. *If $(S_n;\, n \geqslant 0)$ is a random walk with $\mathbb{E}[S_1] = 0$ and $0 < \mathbb{E}[S_1^2] < \infty$, there exists $D < \infty$ such that, for any $\theta > 0$, $x \geqslant 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_x[\mathrm{e}^{-\theta S_n};\, S_1, \ldots, S_n \geqslant 0] \leqslant D\frac{(1+x)}{(1-\mathrm{e}^{-\theta})^2}\, n^{-3/2}.$$

Lemma 5. *There exists $c_3 < \infty$ such that, for any $t > 0$,*

$$\hat{\mathbb{E}}\left[\mathrm{e}^{-\alpha X_t}\left(\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s\right)^{-\beta}\right] \leqslant c_3\, t^{-3/2}.$$

Proof. Denote by Q_t the left hand side above. The inequality $\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s \geqslant t\,\mathrm{e}^{-M_t}$ implies that, for any $t \in (0,1]$,

$$t^{3/2}Q_t \leqslant \hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_t + \beta M_t}\big]t^{3/2-\beta} \leqslant \gamma^{-t}\,\mathbb{E}\big[\mathrm{e}^{\beta M_t}\big] \leqslant \gamma^{-1}\,\mathbb{E}\big[\mathrm{e}^{\beta M_1}\big].$$

The last term is finite by Lemma 1. We shall prove that $\sup_{t>1} t^{3/2}Q_t < \infty$. Let $n \in \mathbb{N}$, and $Z_j = \log\big(\int_j^{j+1} \mathrm{e}^{-(X_s - X_j)}\,\mathrm{d}s\big)$. Then we see

$$\int_0^{n+1} \mathrm{e}^{-X(s)}\,\mathrm{d}s = \sum_{j=0}^{n} \mathrm{e}^{-X(j)+Z(j)} \geqslant \mathrm{e}^{-X(\rho)+Z(\rho)},$$

where $\rho = \min\{k \leqslant n : X_k = \min_{0\leqslant j\leqslant n} X_j\}$. Hence

$$Q_{n+1} \leqslant \sum_{j=0}^{n} \hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_{n+1}+\beta(X_j - Z_j)};\ \rho = j\big] =: \sum_{j=0}^{n} l_j.$$

If $1 \leqslant j \leqslant n-1$, by the duality of the walk $(X_k;\ 0 \leqslant k \leqslant n)$ and Lemma 4,

$$\begin{aligned} l_j &= \hat{\mathbb{E}}\big[\mathrm{e}^{-(\beta-\alpha)\bar{X}_j};\ \bar{X}_1,\dots,\bar{X}_j > 0\big]\hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_{n+1-j}-\beta Z_0};\ X_1,\dots,X_{n-j} \geqslant 0\big] \\ &\leqslant d_1 j^{-3/2} \times \gamma^{-1}\hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_{n-j}-\beta Z_0};\ X_1,\dots,X_{n-j} \geqslant 0\big], \end{aligned}$$

with some $d_1 < \infty$. (In this proof d_i denotes a certain positive constant.) We estimate the expectation in the last term. Using first the Markov property, and then Lemma 4, we have that, if $n \geqslant 2$,

$$\begin{aligned} &\hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_n-\beta Z_0};\ X_1,\dots,X_n \geqslant 0\big] \\ &\qquad = \hat{\mathbb{E}}\Big[\mathrm{e}^{-\beta Z_0} 1_{(X_1\geqslant 0)} \hat{\mathbb{E}}_{X_1}\big[\mathrm{e}^{-\alpha X_{n-1}};\ X_1,\dots,X_{n-1} \geqslant 0\big]\Big] \\ &\qquad \leqslant d_2\, n^{-3/2}\, \hat{\mathbb{E}}\big[\mathrm{e}^{-\beta Z_0}(1+|X_1|)\big] \\ &\qquad = d_3\, n^{-3/2}. \end{aligned}$$

Because of Lemma 3, d_3 is finite. If $n = 1$, by Lemma 1, $\hat{\mathbb{E}}[\mathrm{e}^{-\alpha X_1-\beta Z_0}] \leqslant \gamma^{-1}\,\mathbb{E}[\mathrm{e}^{\beta M_1}] < \infty$. Combining these estimates, we get

$$l_j \leqslant d_4\, j^{-3/2}(n-j)^{-3/2} \qquad \text{if } 1 \leqslant j \leqslant n-1.$$

The similar calculations show that

$$\begin{aligned} l_0 &= \gamma^{-1}\hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_n-\beta Z_0};\ X_1,\dots,X_n \geqslant 0\big] \leqslant d_5\, n^{-3/2}, \\ l_n &= \hat{\mathbb{E}}\big[\mathrm{e}^{-(\beta-\alpha)\bar{X}_n};\ \bar{X}_1,\dots,\bar{X}_n > 0\big]\hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_1-\beta Z_0}\big] \leqslant d_6\, n^{-3/2}. \end{aligned}$$

Therefore we have the following.

$$\begin{aligned} Q_{n+1} &\leqslant d_7\, n^{-3/2} + d_8 \sum_{j=1}^{n-1} j^{-3/2}(n-j)^{-3/2} \\ &\leqslant d_7\, n^{-3/2} + 2d_8 \sum_{j=1}^{[n/2]} j^{-3/2}(n-j)^{-3/2} \\ &\leqslant d_7\, n^{-3/2} + 2d_8 \Big(\frac{n}{2}\Big)^{-3/2} \sum_{j=1}^{\infty} j^{-3/2} \\ &= d_9\, n^{-3/2}. \end{aligned}$$

Let $t > 1$, and $n = [t]$. Then we have

$$\begin{aligned} Q_t &\leqslant \hat{\mathbb{E}} \left[\mathrm{e}^{-\alpha X_t} \left(\int_0^n \mathrm{e}^{-X_s} \, \mathrm{d}s \right)^{-\beta} \right] \\ &= \hat{\mathbb{E}} \left[\mathrm{e}^{-\alpha X_{t-n}} \right] Q_n \\ &\leqslant \gamma^{-1} d_{10} \, n^{-3/2} \\ &\leqslant d_{11} \, t^{-3/2}. \end{aligned}$$

This concludes the proof of the lemma. □

Recall (2). Using Lemma 2, the Girsanov transform and Lemma 5 in turn, we get, for any $t > 0$,

$$\mathcal{P}(t) \leqslant c_2 \, \mathbb{E}\left[A_t^{-\beta} \right] = c_2 \gamma^t \hat{\mathbb{E}} \left[\mathrm{e}^{-\alpha X_t} \left(\int_0^t \mathrm{e}^{-X_s} \, \mathrm{d}s \right)^{-\beta} \right] \leqslant c_2 c_3 \, \gamma^t t^{-3/2}.$$

Proposition 1 is proved.

3 The Lower Bound

On Lemma 6, 8 and 9 below we assume that X is in Class I.

Lemma 6. *If* $k > 0$, *then as* $t \to \infty$

$$\mathbb{P}\{\sigma_k > t\} \sim c \, d \, \mathrm{e}^{\alpha k} \bar{U}(k) \gamma^t t^{-3/2},$$

where $c = \int_0^\infty \mathrm{e}^{-\alpha x} U(x) \, \mathrm{d}x$ *and* $d = c_1 \big/ \sqrt{2\pi |\phi''(\alpha)|}$.

Proof. It is easy to see that $\gamma^{-t} \mathrm{e}^{-\alpha k} \, \mathbb{P}\{\sigma_k > t\} = \hat{\mathbb{E}}_k[\mathrm{e}^{-\alpha X_t}; \, \sigma_0 > t]$. According to [10, Lemma 1], the right hand side is of the order $c \, d \, \bar{U}(k) t^{-3/2}$ as $t \to \infty$. The lemma is proved. □

Lemma 7. *There exists* $c_4 < \infty$ *such that, for any* $t > 0$, $k > 0$ *and* $x > -k$,

$$\mathbb{P}_x\{\sigma_k > t\} \leqslant c_4 \mathrm{e}^{\alpha(x+k)} (1 + x + k) \gamma^t t^{-3/2}.$$

Proof. We only have to show the lemma if $t > 1$. Put $y = x + k$ and $n = [t]$. Then, by Lemma 4,

$$\begin{aligned} \mathbb{P}_x\{\sigma_k > t\} = \mathbb{P}_y\{\sigma_0 > t\} &\leqslant \mathbb{P}_y\{X_1 > 0, \ldots, X_n > 0\} \\ &= \gamma^n \mathrm{e}^{\alpha y} \hat{\mathbb{E}}_y \left[\mathrm{e}^{-\alpha X_n}; \, X_1 > 0, \ldots, X_n > 0 \right] \\ &\leqslant \text{const.} \gamma^n \mathrm{e}^{\alpha y} (1 + y) n^{-3/2}, \end{aligned}$$

which shows the lemma. □

Lemma 8. *Let $k > 0$ and $F : \mathbb{D}[0,s] \to \mathbb{R}$ be continuous and bounded. Then as $t \to \infty$,*

$$\begin{aligned}
&\mathbb{E}\big[F(X_u;\, u \leqslant s)F(X_{(t-u)-};\, u \leqslant s) \,\big|\, \sigma_k > t\big] \\
&\quad \longrightarrow \mathbb{E}_k\big[F(Y_u - k;\, u \leqslant s)\big]\, \frac{1}{c} \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\alpha z} U(z)\, \mathbb{E}_z\big[F(\bar{Y}_u - k;\, u \leqslant s)\big].
\end{aligned}$$

The preceding lemma can be derived from [10, Theorem 2]. Using Lemmas 6, 7 and 8, we get the following.

Lemma 9. *If $\lambda > 0$ and $k > 0$, then*

$$\lim_{t\to\infty} \gamma^{-t} t^{3/2}\, \mathbb{E}\big[\mathrm{e}^{-\lambda A_t};\, \sigma_k > t\big] = J_k(\lambda),$$

where $J_k(\lambda)$ is equal to

$$\begin{aligned}
&d\, \mathrm{e}^{\alpha k} \bar{U}(k)\, \mathbb{E}_k \exp\Big(-\lambda\, \mathrm{e}^k \int_0^\infty \mathrm{e}^{-Y_s}\, \mathrm{d}s\Big) \\
&\qquad\qquad \times \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\alpha z} U(z)\, \mathbb{E}_z \exp\Big(-\lambda\, \mathrm{e}^k \int_0^\infty \mathrm{e}^{-\bar{Y}_s}\, \mathrm{d}s\Big).
\end{aligned}$$

Proof. For the function $F(\omega) = \exp\big(-\lambda \int_0^s \mathrm{e}^{-\omega(u)}\, \mathrm{d}u\big)$, $\omega \in \mathbb{D}[0,s]$, we can use Lemma 8. Applying Lemmas 6 and 8 with the function F, we have

$$\lim_{s\to\infty} \lim_{t\to\infty} \gamma^{-t} t^{3/2}\, \mathbb{E}\left[\exp\Big(-\lambda \int_{[0,s]\cup[t-s,t]} \mathrm{e}^{-X_u}\, \mathrm{d}u\Big);\, \sigma_k > t\right] = J_k(\lambda).$$

Recall that $A_t = \int_0^t \mathrm{e}^{-X_u}\, \mathrm{d}u$. Then

$$0 \leqslant \exp\Big(-\lambda \int_{[0,s]\cup[t-s,t]} \mathrm{e}^{-X_u}\, \mathrm{d}u\Big) - \exp(-\lambda A_t) \leqslant \lambda \int_s^{t-s} \mathrm{e}^{-X_u}\, \mathrm{d}u.$$

By these estimates, our lemma follows from

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \gamma^{-t} t^{3/2} \int_s^{t-s} \mathbb{E}\big[\mathrm{e}^{-X_u};\, \sigma_k > t\big]\, \mathrm{d}u = 0.$$

We show the above. Using Lemma 7, we see that, for any $x > -k$,

$$\begin{aligned}
\mathrm{e}^{-x}\, \mathbb{P}_x\{\sigma_k > t\} &\leqslant c_4 \mathrm{e}^k \sup_{z\geqslant 0}\big\{(1+z)\, \mathrm{e}^{-(1-\alpha)z}\big\} \gamma^t t^{-3/2} \\
&= \text{const.}\, \mathrm{e}^k \gamma^t t^{-3/2}.
\end{aligned}$$

Applying first the Markov property, and then the inequality above, we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-X_u};\, \sigma_k > t] &= \mathbb{E}\big[\mathrm{e}^{-X_u}\, \mathbb{P}_{X_u}\{\sigma_k > t-u\};\, \sigma_k > u\big] \\
&\leqslant \text{const.}\, \mathrm{e}^{2k} \gamma^t (t-u)^{-3/2} u^{-3/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}\int_s^{t-s} \mathbb{E}\big[\mathrm{e}^{-X_u};\,\sigma_k > t\big]\,\mathrm{d}u &\leqslant \mathrm{const.}\,\mathrm{e}^{2k}\gamma^t \int_s^{t-s} (t-u)^{-3/2}u^{-3/2}\,\mathrm{d}u \\ &= \mathrm{const.}\,\mathrm{e}^{2k}\gamma^t \int_s^{t/2} (t-u)^{-3/2}u^{-3/2}\,\mathrm{d}u \\ &\leqslant \mathrm{const.}\,\mathrm{e}^{2k}\gamma^t t^{-3/2}s^{-1/2},\end{aligned}$$

which shows the desired result. Hence we get the lemma. □

The positivity of $J_k(\lambda)$, which we use in the proof of Proposition 2, follows from the next lemma.

Lemma 10. *For any $x > 0$,*

$$\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-Y_t}\,\mathrm{d}t\right] \qquad \textit{and} \qquad \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\bar{Y}_t}\,\mathrm{d}t\right]$$

are not greater than $3\,c_1U(1)\bar{U}(1)$.

Proof. We prove only the claim to Y. By Fubini's theorem and the definition of Y,

$$\bar{U}(x)\,\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-Y_t}\,\mathrm{d}t\right] = \hat{\mathbb{E}}_x\left[\int_0^{\sigma_0} \mathrm{e}^{-X_t}\bar{U}(X_t)\,\mathrm{d}t\right].$$

The right hand side is written as follows: see e.g. [2, p.176] or [10, Lemma 10].

$$\hat{\mathbb{E}}_x\left[\int_0^{\sigma_0} \mathrm{e}^{-X_t}\bar{U}(X_t)\,\mathrm{d}t\right] = c_1\int_0^\infty \mathrm{d}\mathcal{V}(y)\int_{[0,x)} \mathrm{d}\bar{\mathcal{V}}(z)\,\mathrm{e}^{-(x+y-z)}\bar{U}(x+y-z),$$

where $\mathcal{V}(\,\cdot\,) = U(\,\cdot\,+)$ and $\bar{\mathcal{V}}(\,\cdot\,) = \bar{U}(\,\cdot\,+)$. Using the inequality $\bar{U}(x) \leqslant \bar{U}(1)(x+1)$ and the integration by parts formula, we have

$$\begin{aligned}\int_{[0,x)} \mathrm{e}^{z}\bar{U}(x+y-z)\,\mathrm{d}\bar{\mathcal{V}}(z) &\leqslant \bar{U}(1)\int_{[0,x)} \mathrm{e}^{z}(x+y-z+1)\,\mathrm{d}\bar{\mathcal{V}}(z) \\ &\leqslant \bar{U}(1)\bar{U}(x)\,\mathrm{e}^{x}(y+1).\end{aligned}$$

In the same way,

$$\begin{aligned}\hat{\mathbb{E}}_x\left[\int_0^{\sigma_0} \mathrm{e}^{-X_t}\bar{U}(X_t)\,\mathrm{d}t\right] &\leqslant c_1\bar{U}(1)\bar{U}(x)\int_0^\infty \mathrm{e}^{-y}(y+1)\,\mathrm{d}\mathcal{V}(y) \\ &\leqslant c_1\bar{U}(1)\bar{U}(x)U(1)\int_0^\infty \mathrm{e}^{-y}y(y+1)\,\mathrm{d}y \\ &\leqslant 3\,c_1\bar{U}(1)\bar{U}(x)U(1).\end{aligned}$$

This combined with the first equation in this proof shows our claim. □

By Jensen's inequality and Lemma 10, we observe that, for all $\lambda > 0$,

$$\begin{aligned} J_k(\lambda) &\geqslant d\,\mathrm{e}^{\alpha k}\bar{U}(k)\exp\biggl(-\lambda\,\mathrm{e}^k\,\mathbb{E}_k\biggl[\int_0^\infty \mathrm{e}^{-Y_s}\,\mathrm{d}s\biggr]\biggr) \\ &\qquad\times \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-\alpha z}U(z)\exp\biggl(-\lambda\,\mathrm{e}^k\,\mathbb{E}_z\biggl[\int_0^\infty \mathrm{e}^{-\bar{Y}_s}\,\mathrm{d}s\biggr]\biggr) \\ &\geqslant d\,\mathrm{e}^{\alpha k}\bar{U}(k)\int_0^\infty \mathrm{e}^{-\alpha z}U(z)\,\mathrm{d}z \times \exp\bigl(-6c_1\lambda\,\mathrm{e}^kU(1)\bar{U}(1)\bigr) \\ &> 0. \end{aligned}$$

Obviously $J_k(\lambda)$ is non-decreasing in k. Therefore there exists a positive limit $J_\infty(\lambda) := \lim_{k\to\infty} J_k(\lambda)$. Recall (2) and rewrite $f(x) = \int_0^\infty \mathrm{e}^{-x\lambda}\,\mathbb{E}[A\,\mathrm{e}^{-\lambda A}]\,\mathrm{d}\lambda$. Then we have

$$\mathcal{P}(t) \geqslant \int_0^\infty \mathbb{E}\bigl[\mathrm{e}^{-\lambda A_t};\, \sigma_k > t\bigr]\,\mathbb{E}\bigl[A\,\mathrm{e}^{-\lambda A}\bigr]\,\mathrm{d}\lambda.$$

Using first Lemma 9 with Fatou's lemma, and then the monotone convergence theorem in k, we get

$$\liminf_{t\to\infty} \gamma^{-t}t^{3/2}\mathcal{P}(t) \geqslant \int_0^\infty J_\infty(\lambda)\,\mathbb{E}\bigl[A\,\mathrm{e}^{-\lambda A}\bigr]\,\mathrm{d}\lambda =: C > 0. \tag{3}$$

The positivity of C comes from the fact that $J_\infty(\lambda) > 0$ and $\mathbb{E}[A\,\mathrm{e}^{-\lambda A}] > 0$ for $\lambda > 0$. We investigate the structure of $J_\infty(\lambda)$ (especially for the convenience of Section 5). By the change of variable $x = z - k$, $J_k(\lambda)$ is expressed as

$$J_k(\lambda) = \frac{c_1}{\sqrt{2\pi|\phi''(\alpha)|}}\,g_{\lambda.k}(0)\int_{-k}^\infty \mathrm{e}^{-\alpha x}\bar{g}_{\lambda.k}(x)\,\mathrm{d}x, \tag{4}$$

where

$$\begin{aligned} g_{\lambda.k}(x) &= \bar{U}(k+x)\,\mathbb{E}_{k+x}\biggl[\exp\biggl(-\lambda\,\mathrm{e}^k\int_0^\infty \mathrm{e}^{-Y_s}\,\mathrm{d}s\biggr)\biggr], \qquad x > -k, \\ \bar{g}_{\lambda.k}(x) &= U(k+x)\,\mathbb{E}_{k+x}\biggl[\exp\biggl(-\lambda\,\mathrm{e}^k\int_0^\infty \mathrm{e}^{-\bar{Y}_s}\,\mathrm{d}s\biggr)\biggr], \qquad x > -k. \end{aligned}$$

Recalling the definition of Y, we see

$$\begin{aligned} g_{\lambda.k}(x) &= \lim_{t\to\infty} \bar{U}(k+x)\,\mathbb{E}_{k+x}\biggl[\exp\biggl(-\lambda\,\mathrm{e}^k\int_0^t \mathrm{e}^{-Y_s}\,\mathrm{d}s\biggr)\biggr] \\ &= \lim_{t\to\infty} \hat{\mathbb{E}}_{k+x}\biggl[\exp\biggl(-\lambda\,\mathrm{e}^k\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s\biggr)\bar{U}(X_t);\, \sigma_0 > t\biggr] \\ &= \lim_{t\to\infty} \hat{\mathbb{E}}_x\biggl[\exp\biggl(-\lambda\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s\biggr)\bar{U}(k+X_t);\, \sigma_k > t\biggr]. \end{aligned}$$

The expectation in the last term is non-decreasing in k, so is $g_{\lambda.k}(x)$. Hence we can define the following limits for each $x \in \mathbb{R}$ and $\lambda > 0$.

$$\begin{aligned} g_\lambda(x) &:= \lim_{k\to\infty} \bar{U}(k+x)\,\mathbb{E}_{k+x}\left[\exp\left(-\lambda\,\mathrm{e}^k\int_0^\infty \mathrm{e}^{-Y_s}\,\mathrm{d}s\right)\right], \\ \bar{g}_\lambda(x) &:= \lim_{k\to\infty} U(k+x)\,\mathbb{E}_{k+x}\left[\exp\left(-\lambda\,\mathrm{e}^k\int_0^\infty \mathrm{e}^{-\bar{Y}_s}\,\mathrm{d}s\right)\right]. \end{aligned} \tag{5}$$

Letting $k \to \infty$ in (4), by the monotone convergence theorem and (5), we have

$$J_\infty(\lambda) = \frac{c_1}{\sqrt{2\pi|\phi''(\alpha)|}}\, g_\lambda(0)\int_{-\infty}^\infty \mathrm{e}^{-\alpha x}\bar{g}_\lambda(x)\,\mathrm{d}x. \tag{6}$$

The combination of (3), (5) and (6) establishes Proposition 2.

4 Proof of Theorem 1

The results in the previous sections enable us to prove Theorem 1. Propositions 1 and 2 ensure $0 < C < \infty$, so that Theorem 1 follows from the estimate $\limsup_{t\to\infty} \gamma^{-t}t^{3/2}\mathcal{P}(t) \leqslant C$. We show it. Recall (2). Since $f(x)$ is decreasing, for any $\delta > 0$ and $k > 0$,

$$\begin{aligned} \mathcal{P}(t) &= \mathbb{E}[f(A_t);\, \sigma_k > t-\delta] + \mathbb{E}[f(A_t);\, \sigma_k \leqslant t-\delta] \\ &\leqslant \mathbb{E}[f(A_{t-\delta});\, \sigma_k > t-\delta] + \mathbb{E}[f(A_t);\, \sigma_k \leqslant t-\delta]. \end{aligned} \tag{7}$$

Thanks to the expression $f(x) = \int_0^\infty \mathrm{e}^{-x\lambda}\,\mathbb{E}[A\,\mathrm{e}^{-\lambda A}]\,\mathrm{d}\lambda$, for any $s > 0$,

$$\mathbb{E}[f(A_s);\, \sigma_k > s] = \int_0^\infty \mathbb{E}\big[\mathrm{e}^{-\lambda A_s};\, \sigma_k > s\big]\,\mathbb{E}\big[A\,\mathrm{e}^{-\lambda A}\big]\,\mathrm{d}\lambda.$$

Plainly $\mathbb{E}[\mathrm{e}^{-\lambda A_s};\, \sigma_k > s] \leqslant \mathbb{P}\{\sigma_k > s\}$ and $\int_0^\infty \mathbb{E}[A\,\mathrm{e}^{-\lambda A}]\,\mathrm{d}\lambda = 1$. Thus, by Lemmas 6 and 9 with the dominated convergence theorem,

$$\lim_{t\to\infty} \gamma^{-t}t^{3/2}\,\mathbb{E}[f(A_{t-\delta});\, \sigma_k > t-\delta] = \gamma^{-\delta}\int_0^\infty J_k(\lambda)\,\mathbb{E}\big[A\,\mathrm{e}^{-\lambda A}\big]\mathrm{d}\lambda. \tag{8}$$

Using Lemma 2, we have

$$\begin{aligned} \gamma^{-t}\,\mathbb{E}[f(A_t);\, \sigma_k \leqslant t-\delta] &= \hat{\mathbb{E}}\big[\mathrm{e}^{-\alpha X_t}f(A_t);\, \sigma_k \leqslant t-\delta\big] \\ &\leqslant c_2\hat{\mathbb{E}}\left[\mathrm{e}^{-\alpha X_t}\left(\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s\right)^{-\beta};\, \sigma_k \leqslant t-\delta\right] \\ &\leqslant c_2\hat{\mathbb{E}}\left[\mathrm{e}^{-\alpha X_t}\left(\int_{\sigma_k}^t \mathrm{e}^{-X_s}\,\mathrm{d}s\right)^{-\beta};\, \sigma_k \leqslant t-\delta\right] \\ &\leqslant c_2c_3\mathrm{e}^{-(\beta-\alpha)k}\hat{\mathbb{E}}[(t-\sigma_k)^{-3/2};\, \sigma_k \leqslant t-\delta]. \end{aligned} \tag{9}$$

The last inequality comes from the strong Markov property conditioning on $\mathcal{F}_{\sigma_k}$ and Lemma 5. From the Girsanov transform $\hat{\mathbb{P}}$ on $\mathcal{F}_{\sigma_k}$,

$$\begin{aligned}\hat{\mathbb{P}}\{t < \sigma_k \leqslant t+u\} &= \mathbb{E}\big[\mathrm{e}^{\alpha X(\sigma_k)+\sigma_k\phi(\alpha)};\, t < \sigma_k \leqslant t+u\big]\\ &\leqslant \gamma^{-(t+u)}\mathrm{e}^{-\alpha k}\,\mathbb{P}\{\sigma_k > t\}\\ &\leqslant c_4\gamma^{-u}(1+k)t^{-3/2}.\end{aligned}$$

In the last inequality we used Lemma 7. Let $n = [t]$ and $\delta < 1$. Applying the estimate above in the first inequality below, we obtain

$$\begin{aligned}&\hat{\mathbb{E}}[(t-\sigma_k)^{-3/2};\, \sigma_k \leqslant t-\delta]\\ &= \sum_{j=1}^{n-1}\hat{\mathbb{E}}\big[(t-\sigma_k)^{-3/2};\, j-1 < \sigma_k \leqslant j\big] + \hat{\mathbb{E}}\big[(t-\sigma_k)^{-3/2};\, n-1 < \sigma_k \leqslant t-\delta\big]\\ &\leqslant c_5(1+k)\Bigg(\sum_{j=1}^{n-1}(n-j)^{-3/2}j^{-3/2} + \delta^{-3/2}n^{-3/2}\Bigg)\\ &\leqslant c_6(1+k)\delta^{-3/2}n^{-3/2}\\ &\leqslant c_7(1+k)\delta^{-3/2}t^{-3/2}.\end{aligned}$$

Combining (9) with the above, we observe

$$\limsup_{t\to\infty}\gamma^{-t}t^{3/2}\,\mathbb{E}[f(A_t);\, \sigma_k \leqslant t-\delta] \leqslant c_8\delta^{-3/2}(1+k)\,\mathrm{e}^{-(\beta-\alpha)k}.$$

In view of (7), (8) and the preceding inequality, we get

$$\limsup_{t\to\infty}\gamma^{-t}t^{3/2}\mathcal{P}(t) \leqslant \gamma^{-\delta}\int_0^\infty J_k(\lambda)\,\mathbb{E}\big[A\,\mathrm{e}^{-\lambda A}\big]\,\mathrm{d}\lambda + c_8\delta^{-3/2}(1+k)\,\mathrm{e}^{-(\beta-\alpha)k}.$$

Letting $k \to \infty$, and then $\delta \downarrow 0$, we see that the right hand side above tends to C (cf. (3)). This concludes the proof of the theorem.

5 The Drifted Brownian Case

In this section we compute the precise value of C in case of the drifted Brownian environment. Let $X_t = B_t - \alpha t$ with $0 < \alpha < 1$ and $\xi_t = W_t - bt$ with $b > \alpha/2$ where B and W are independent Brownian motions. It is easy to see that the conditions (c)–(e) are fulfilled, and $\gamma = \mathrm{e}^{-\alpha^2/2}$. Moreover X and $\bar{X}$ are Brownian motions under $\hat{\mathbb{P}}$, so that $U(x) = \bar{U}(x) = \sqrt{2}x$ (by our normalization of local times) and $g_\lambda(x) = \bar{g}_\lambda(x)$. In particular Y and $\bar{Y}$ are three-dimensional Bessel processes. To determine $g_\lambda(x)$, we need the following.

Lemma 11. *Let $(R_t;\, t \geqslant 0)$ be a three-dimensional Bessel process. Then, for any $x > 0$ and $\lambda > 0$,*

$$\begin{aligned}\mathbb{E}_x \exp\Big(-\lambda\int_0^\infty \mathrm{e}^{-R_t}\,\mathrm{d}t\Big)\\ &= \frac{2}{x}\bigg(K_0\big(2\sqrt{2\lambda}\,\mathrm{e}^{-x/2}\big) - \frac{K_0\big(2\sqrt{2\lambda}\big)}{I_0\big(2\sqrt{2\lambda}\big)} I_0\big(2\sqrt{2\lambda}\,\mathrm{e}^{-x/2}\big)\bigg),\end{aligned}$$

where I_0 and K_0 are the modified Bessel functions with index 0 of the first and third kind respectively.

Remark 1. In particular, letting $x \to 0$ with L'Hospital's rule, we recover

$$\begin{aligned}\mathbb{E}\exp\Big(-\lambda\int_0^\infty \mathrm{e}^{-R_t}\,\mathrm{d}t\Big) &= -2\sqrt{2\lambda}\bigg(\frac{(K_0' I_0 - K_0 I_0')\big(2\sqrt{2\lambda}\big)}{I_0\big(2\sqrt{2\lambda}\big)}\bigg)\\ &= \frac{1}{I_0\big(2\sqrt{2\lambda}\big)}.\end{aligned}$$

In the second equality we used the Wronskian relation $(K_0 I_0' - K_0' I_0)(y) = 1/y$. This formula is in agreement with Donati–Martin and Yor [7].

Proof of Lemma 11. According to the formula 2.10.1 in [4, p. 345], we have that, if $z \geqslant x$,

$$\mathbb{E}_x \exp\Big(-\lambda\int_0^{T_z} \mathrm{e}^{-R_t}\,\mathrm{d}t\Big) = \frac{z\,S_0\big(2\sqrt{2\lambda}\,\mathrm{e}^{-x/2}, 2\sqrt{2\lambda}\big)}{x\,S_0\big(2\sqrt{2\lambda}\,\mathrm{e}^{-z/2}, 2\sqrt{2\lambda}\big)},$$

where $T_z = \inf\{t \geqslant 0 : R_t = z\}$ and $S_0(a,b) = I_0(a)K_0(b) - K_0(a)I_0(b)$. Recall that $I_0(a) \to 1$ and $K_0(a) \sim -\log a$ as $a \to 0$. Letting $z \to \infty$ in the equality above, we get the lemma. □

Since $K_0(a) \to 0$ and $I_0(a) \to \infty$ as $a \to \infty$, by (5) and Lemma 11, we have $g_\lambda(x) = 2^{3/2} K_0(2\sqrt{2\lambda}\,\mathrm{e}^{-x/2})$. Hence, by (6),

$$\begin{aligned}J_\infty(\lambda) &= \frac{2^{5/2}}{\sqrt{\pi}} K_0\big(2\sqrt{2\lambda}\big)\int_{-\infty}^\infty \mathrm{e}^{-\alpha y} K_0\big(2\sqrt{2\lambda}\,\mathrm{e}^{-y/2}\big)\,\mathrm{d}y\\ &= \frac{2^{7/2}}{\sqrt{\pi}} K_0\big(2\sqrt{2\lambda}\big)\int_0^\infty z^{2\alpha-1} K_0\big(2\sqrt{2\lambda} z\big)\,\mathrm{d}z \qquad (10)\\ &= \frac{2^{3/2-\alpha}}{\sqrt{\pi}}\,\Gamma(\alpha)^2 \lambda^{-\alpha} K_0\big(2\sqrt{2\lambda}\big).\end{aligned}$$

In the third equality we used the identity $\int_0^\infty t^{2\nu-1} K_0(t)\,\mathrm{d}t = 4^{\nu-1}\Gamma(\nu)^2$, $\nu > 0$. The distribution of A is given by the following result due to Dufresne [8]. We also refer to Yor [14].

Lemma 12. *For any $\kappa > 0$, we have*

$$\int_0^\infty \mathrm{d}t\,\exp\Big(W(t) - \frac{\kappa}{2}t\Big) \stackrel{\mathrm{d}}{=} \frac{2}{Z_\kappa},$$

where $\stackrel{\mathrm{d}}{=}$ *means equality in law and* Z_κ *is a gamma variable of index* κ*, i.e.,*

$$\mathbb{P}\{Z_\kappa \in \mathrm{d}t\} = \frac{t^{\kappa-1}\mathrm{e}^{-t}}{\Gamma(\kappa)}\,\mathrm{d}t, \qquad t > 0.$$

By Lemma 12, $A \stackrel{\mathrm{d}}{=} 2/Z_{2b}$. In other words

$$\mathbb{P}\{A \in \mathrm{d}x\} = \frac{2^{2b}}{\Gamma(2b)}\, x^{-(2b+1)}\, \mathrm{e}^{-2/x}\, \mathrm{d}x, \qquad x > 0.$$

Thus (3) and (10) combined with the above tell us that

$$C = \frac{2^{2b-\alpha+3/2}}{\sqrt{\pi}} \times \frac{\Gamma(\alpha)^2}{\Gamma(2b)} \int_0^\infty\!\!\int_0^\infty \lambda^{-\alpha} x^{-2b} \mathrm{e}^{-(\lambda x + 2/x)} K_0(2\sqrt{2\lambda})\,\mathrm{d}x\,\mathrm{d}\lambda.$$

If we use the identity

$$K_0(z) = \int_0^\infty \exp\Big(-t - \frac{z^2}{4\,t}\Big)\frac{\mathrm{d}t}{2t}, \qquad z > 0,$$

the double integral in C is written as follows.

$$\begin{aligned}
&\int_0^\infty\!\!\int_0^\infty \mathrm{d}x\,\mathrm{d}t\,(2t)^{-1}x^{-2b}\mathrm{e}^{-(t+2/x)}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-\alpha}\mathrm{e}^{-(x+2/t)\lambda}\\
&\quad = \Gamma(1-\alpha)\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}t\,(2t)^{-1}x^{-2b}\mathrm{e}^{-(t+2/x)}(x+2/t)^{\alpha-1}\\
&\quad = \frac{\Gamma(1-\alpha)}{2^{2b-\alpha+1}}\int_0^\infty \mathrm{d}y\int_0^\infty \mathrm{d}t\,(1+y)^{\alpha-1}(ty)^{2b-\alpha-1}\mathrm{e}^{-(1+y)t} \qquad (x = 2/ty)\\
&\quad = \frac{1}{2^{2b-\alpha+1}}\Gamma(1-\alpha)\Gamma(2b-\alpha)\int_0^\infty y^{2b-\alpha-1}(1+y)^{2\alpha-2b-1}\,\mathrm{d}y\\
&\quad = \frac{1}{2^{2b-\alpha+1}}\Gamma(1-\alpha)\Gamma(2b-\alpha)\mathrm{B}(1-\alpha, 2b-\alpha).
\end{aligned}$$

Consequently we get the following.

Proposition 3. *Assume that* $X_t = B_t - \alpha t$ *with* $0 < \alpha < 1$*, and* $\xi_t = W_t - bt$ *with* $b > \alpha/2$*. Then as* $x \to \infty$*,*

$$\mathcal{P}(x) \sim Cx^{-3/2}\exp(-x\alpha^2/2)$$

where

$$C = \frac{(2\pi)^{3/2}}{1-\cos(2\pi\alpha)} \times \frac{\Gamma(2b-\alpha)^2}{\Gamma(2b)\Gamma(2b-2\alpha+1)}.$$

When $b = \alpha$, the same asymptotic was first obtained by Kawazu–Tanaka [11], and also appears in Comtet–Monthus–Yor [6]. In [11], they say

$$C = \frac{2^{5/2-2\alpha}}{\sqrt{\pi}\,\Gamma(2\alpha)} \int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty \frac{x}{x+y}\, y^{2\alpha-1} z^{2\alpha} \mathrm{e}^{-(y/2+\nu x)} u \sinh u \,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where $\nu = (1+z^2)/2 + z\cosh u$. We may check the equivalence of the two expressions of C in case of $b = \alpha$. To see it, we use the following.

$$2K_0(x)K_0(z) = \int_0^\infty \exp\Big(-\frac{v}{2} - \frac{x^2+z^2}{2v}\Big) K_0\Big(\frac{xz}{v}\Big)\frac{\mathrm{d}v}{v}, \qquad x, z > 0,$$

$$K_0(y) = \int_0^\infty \mathrm{e}^{-y\cosh u}\,\mathrm{d}u = y\int_0^\infty \mathrm{e}^{-y\cosh u} u \sinh u\,\mathrm{d}u, \qquad y > 0.$$

Go back to (10). These formulae and the change of variable $v = 8\lambda/x$ imply

$$\begin{aligned}
2^{-5/2}\sqrt{\pi} J_\infty(\lambda) &= \int_0^\infty z^{2\alpha-1} 2K_0(2\sqrt{2\lambda})K_0(2\sqrt{2\lambda}z)\,\mathrm{d}z \\
&= \int_0^\infty z^{2\alpha-1}\,\mathrm{d}z \int_0^\infty \exp\Big(-\frac{v}{2} - \frac{8\lambda(1+z^2)}{2v}\Big) K_0\Big(\frac{8\lambda z}{v}\Big)\frac{\mathrm{d}v}{v} \\
&= \int_0^\infty z^{2\alpha-1}\,\mathrm{d}z \int_0^\infty \exp\Big(-\frac{4\,\lambda}{x} - \frac{x(1+z^2)}{2}\Big) K_0(xz)\frac{\mathrm{d}x}{x} \\
&= \int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty z^{2\alpha} \exp(-4\,\lambda/x - \nu x)\, u\sinh u\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}$$

Recall (3). Multiply the above by $\mathbb{E}[A\,\mathrm{e}^{-\lambda A}]$, and integrate over $(0,\infty)$ in λ. Then we may end the computations. Thus all that remains to show is the following. Using the fact $A \overset{\mathrm{d}}{=} 2/Z_{2\alpha}$ by Lemma 12 and Fubini's theorem, we have

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-4\,\lambda/x}\,\mathbb{E}\big[A\,\mathrm{e}^{-\lambda A}\big]\,\mathrm{d}\lambda &= \frac{2^{2\alpha}}{\Gamma(2\alpha)} \int_0^\infty \frac{x}{xw+4}\, w^{-2\alpha}\mathrm{e}^{-2/w} dw \\
&= \frac{2^{-2\alpha}}{\Gamma(2\alpha)} \int_0^\infty \frac{x}{x+y}\, y^{2\alpha-1}\mathrm{e}^{-y/2}\,\mathrm{d}y \qquad (w = 4/y),
\end{aligned}$$

which shows the equivalence of C. Therefore, when $b = \alpha$, Proposition 3 accords with Kawazu–Tanaka [11].

We point out the following. The proof of Kawazu–Tanaka [11] relies essentially upon the formula of the joint distribution $(\mathrm{e}^{B(t)}, \int_0^t \mathrm{e}^{2B(s)}\,\mathrm{d}s)$ for fixed $t > 0$, which was given by Yor [15]. Needless to say, this formula is very useful. However we cannot expect an analogous one if Brownian motion is replaced by a Lévy process. One of our motivations was to get Proposition 3 without such a formula. Such an attempt has been already done by Kotani [12] with analytic methods. The function $g_\lambda(-x) = 2^{3/2}K_0(2\sqrt{2\lambda}\mathrm{e}^{x/2})$ is nothing but $g_\lambda(-\infty, x)$ appeared in [12] where $g_\lambda(x,y)$ is the Green function of $2^{-1}\mathrm{e}^{-x}\Delta$ on $\mathbb{R}$ and $-\infty$ is the entrance boundary of the corresponding diffusion.

6 Appendix

In this appendix we consider 1) and 2) of Theorem 0.

First point

We note that the condition $\phi''(1) < 0$ may be omitted from 2) of Theorem 0. We show it under somewhat mild hypotheses. Suppose that $|\phi(1)| < \infty$, $\phi'(1-) = 0$ and $-\infty \leqslant \mathbb{E}[X_1] < 0$. By the hypothesis $|\phi(1)| < \infty$, for $\theta \in (0,1)$,

$$\mathrm{e}^{-\phi(\theta)}\big(\phi'(\theta)^2 - \phi''(\theta)\big) = \mathbb{E}\big[X_1^2\,\mathrm{e}^{\theta X_1}\big].$$

Since $\phi'(1-) = 0$, letting $\theta \uparrow 1$, we obtain

$$-\mathrm{e}^{-\phi(1)}\phi''(1-) = \mathbb{E}\big[X_1^2\,\mathrm{e}^{X_1}\big] \geqslant 0.$$

If $\phi''(1-) = 0$, the preceding relation yields that $X_1 = 0$ almost surely, so that $\mathbb{E}[X_1] = 0$. It is a contradiction. As a result $\phi''(1-) < 0$, which shows our assertion because $\phi'(1-) = \phi'(1)$ and $\phi''(1-) = \phi''(1)$ under the condition (a).

Second point

The following result is an extension of iv) of Proposition 3.1 in [5]. This lemma will be used in the proof of Proposition B in the third point.

Lemma A. *If* $(\zeta_t;\, t \geqslant 0)$ *is a Lévy process satisfying* $\mathbb{E}[\zeta_1] \geqslant 0$*, then*

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\zeta_t}\,\mathrm{d}t\right]^{-1} = \mathbb{E}[\zeta_1].$$

Proof. Let $\mathbb{E}[\zeta_1] > 0$. The strong law of large numbers states that $\int_0^\infty \mathrm{e}^{-\zeta_s}\,\mathrm{d}s < \infty$ almost surely. Set $H_t = \mathrm{e}^{-\zeta_t}/\int_t^\infty \mathrm{e}^{-\zeta_s}\,\mathrm{d}s$, and define the shift operator $(\theta_t;\, t \geqslant 0)$ such that $\zeta_s(\theta_t\omega) = \zeta_{s+t}(\omega) - \zeta_t(\omega)$. Then

$$H_t(\omega) = \left(\int_0^\infty \mathrm{e}^{\zeta_t(\omega)-\zeta_{s+t}(\omega)}\,\mathrm{d}s\right)^{-1} = \left(\int_0^\infty \mathrm{e}^{-\zeta_s(\theta_t\omega)}\,\mathrm{d}s\right)^{-1} = H_0(\theta_t\omega).$$

The right derivative of $-\log\big(\int_t^\infty \mathrm{e}^{-\zeta_s}\,\mathrm{d}s\big)$ is H_t which is right continuous. Thus, integrating $H_s(\omega) = H_0(\theta_s\omega)$ over $[0,t]$, we have, for almost every ω,

$$\log\left(\int_0^\infty \mathrm{e}^{-\zeta_s(\omega)}\,\mathrm{d}s\right) - \log\left(\int_t^\infty \mathrm{e}^{-\zeta_s(\omega)}\,\mathrm{d}s\right) = \int_0^t H_0(\theta_s\omega)\,\mathrm{d}s, \qquad \forall t > 0.$$

Divide both terms by t, and then take the limit as $t \to \infty$. The right (resp. left) hand side converges to $\mathbb{E}[H_0]$ (resp. $\mathbb{E}[\zeta_1]$) by virtue of Birkhoff's ergodic theorem (resp. the strong law of large numbers). Accordingly we get

$$\mathbb{E}[\zeta_1] = \mathbb{E}[H_0] = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\zeta_s}\,\mathrm{d}s\right]^{-1}.$$

Let $\mathbb{E}[\zeta_1] = 0$. Considering $(\zeta_t + \mu t;\, t \geqslant 0)$ with $\mu > 0$, and then letting $\mu \downarrow 0$, we have the desired result. $\square$

Remark 2. Lemma A was also discussed in Bertoin and Yor [3]. They studied the close relation between the distributions of $\int_0^\infty \mathrm{e}^{-\zeta_t}\,\mathrm{d}t$ and the semi-stable Markov process obtained by Lamperti's transform of ζ. See for details [3].

Third point

The following proposition leads to 1) and 2) of Theorem 0, and corresponds to Afanas'ev [1].

Proposition B. *Assume that $\phi(1)$ is finite, and $\psi(1) > 0$.*

1) *If $\phi'(1-) > 0$, then as $x \to \infty$,*

$$\mathcal{P}(x) \sim \mathrm{e}^{-x\phi(1)}\phi'(1-)\psi(1)^{-1}.$$

2) *If $\phi'(1-) = 0$ and $|\phi''(1-)| < \infty$, then*

$$\mathcal{P}(x) \sim \mathrm{e}^{-x\phi(1)}\psi(1)^{-1}\sqrt{|\phi''(1-)|/2\pi x}.$$

We should pay attention to the difference of the conditions between Proposition B and Theorem 0. We mention that the proof of 2) of Theorem 0 depends on the finiteness of $\psi(\theta)$ for some $\theta > 1$, see sect. 4.1. in [5]. Before proving Proposition B, we remark the following. Owing to $\psi(1) > 0$, $\mathbb{E}[A] = \psi(1)^{-1}$ and $xf(x)$ increases to $\mathbb{E}[A]$ as $x \uparrow \infty$. As in Section 1, the Girsanov (or Esscher) transform $\tilde{\mathbb{P}}$ of $\mathbb{P}$ is defined by

$$\tilde{\mathbb{P}} = \mathrm{e}^{X_t + t\phi(1)} \cdot \mathbb{P} \qquad \text{on } \mathcal{F}_t.$$

Then $\mathrm{e}^{t\phi(1)}\mathcal{P}(t) = \tilde{\mathbb{E}}[\mathrm{e}^{-X_t} f(A_t)]$. So the asymptotic of the last term is needed.

Proof. 1) Put $C_t = \int_0^t \mathrm{e}^{X_s}\,\mathrm{d}s$. Using the equivalence in law $(X_s;\, 0 \leqslant s \leqslant t) \overset{\mathrm{d}}{=} (X_t - X_{(t-s)-};\, 0 \leqslant s \leqslant t)$, we have

$$\begin{aligned}\tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} f\big(A_t\big)\big] &= \tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} f(\mathrm{e}^{-X_t}C_t)\big]\\ &= \tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t}C_t f\big(\mathrm{e}^{-X_t}C_t\big)C_t^{-1}\big].\end{aligned}$$

If $t \geqslant 1$, the integrand in the last term is less than $\mathbb{E}[A]C_1^{-1}$. Using the Girsanov transform $\tilde{\mathbb{P}}$ and $(X_s;\, 0 \leqslant s \leqslant 1) \overset{\mathrm{d}}{=} (X_1 - X_{(1-s)-};\, 0 \leqslant s \leqslant 1)$, we see

$$\tilde{\mathbb{E}}\big[C_1^{-1}\big] = \mathrm{e}^{\phi(1)}\,\mathbb{E}\big[A_1^{-1}\big] \leqslant \mathrm{e}^{\phi(1)}\,\mathbb{E}\big[\mathrm{e}^{M_1}\big] < \infty.$$

The finiteness comes from Lemma 1. Note that $\lim_{t\to\infty} \mathrm{e}^{-X_t}C_t = \infty$ $\tilde{\mathbb{P}}$-a.s. because $\tilde{\mathbb{E}}[X_1] = -\phi'(1-) < 0$. Hence, by the dominated convergence theorem,

$$\lim_{t\to\infty} \tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} f(A_t)\big] = \psi(1)^{-1}\tilde{\mathbb{E}}\big[C_\infty^{-1}\big].$$

According to Lemma A, $\tilde{\mathbb{E}}[C_\infty^{-1}] = -\tilde{\mathbb{E}}[X_1] = \phi'(1-)$. Thus 1) is proved.

2) Since $\tilde{\mathbb{E}}[X_1] = -\phi'(1-) = 0$, it was shown in the first point that $\tilde{\mathbb{E}}[X_1^2] = -\phi''(1-) > 0$. Thus, if $\tilde{\mathbb{E}}[X_1^2] = |\phi''(1-)| < \infty$, the following asymptotic holds: see [5, p.99].

$$\tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} A_t^{-1}\big] \sim \sqrt{|\phi''(1-)|/2\pi t} \qquad \text{as } t \to \infty.$$

By Lemma A, if $k > 0$, $\tilde{\mathbb{P}}(A_\infty \leqslant k) \leqslant k\tilde{\mathbb{E}}[A_\infty^{-1}] = k\tilde{\mathbb{E}}[X_1] = 0$. Therefore

$$\begin{aligned}\tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} A_t^{-1};\, A_t \leqslant k\big] &\leqslant \tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t}(A_t - A_{t/2})^{-1};\, A_{t/2} \leqslant k\big] \\ &= \tilde{\mathbb{E}}\big[\mathrm{e}^{-X_{t/2}} A_{t/2}^{-1}\big]\, \tilde{\mathbb{P}}\{A_{t/2} \leqslant k\} \\ &= o\big(t^{-1/2}\big).\end{aligned}$$

Combining the results above, we get, for any fixed $k \geqslant 0$,

$$\tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} A_t^{-1};\, A_t > k\big] \sim \sqrt{|\phi''(1-)|/2\pi t} \qquad \text{as } t \to \infty.$$

Observe that $xf(x) \leqslant \mathbb{E}[A]$ for $\forall x > 0$, and that, for $\forall \varepsilon > 0$, $\exists k > 0$ such that $xf(x) \geqslant (\mathbb{E}[A] - \varepsilon)$ for $\forall x \geqslant k$. Using these inequalities, we have

$$\big(\mathbb{E}[A] - \varepsilon\big)\tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} A_t^{-1};\, A_t > k\big] \leqslant \tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} f(A_t)\big] \leqslant \mathbb{E}[A]\tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} A_t^{-1}\big].$$

The preceding relations show

$$\tilde{\mathbb{E}}\big[\mathrm{e}^{-X_t} f(A_t)\big] \sim \psi(1)^{-1}\sqrt{|\phi''(1-)|/2\pi t} \qquad \text{as } t \to \infty.$$

The proof of the proposition is complete. □

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